Unicity theorems for meromorphic functions

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Abstract: In this paper, we deal with the problem of uniqueness of meromorphic functions sharing three values, and prove some results which are improvements and extensions of many known theorems. Examples are provided to show that the results in this paper are sharp.

Key words: Nevanlinna theory; shared value; unicity theorem.

1. Introduction and main results. In this paper, a meromorphic function means meromorphic in the complex plane. We use the usual notations of Nevanlinna theory of meromorphic functions as explained in [1]. For any nonconstant meromorphic function $f(z)$, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ for $r \to \infty$ except possibly a set of r of finite linear measure. Let k be a positive integer, we denote by $N_{k)}(r, a, f)$ the counting function of a-points of f with multiplicity $\leq k$, and denote by $N_{(k}(r, a, f)$ the counting function of a-points of f with multiplicity$\geq k$ (see [2]).

Let f and g be two nonconstant meromorphic functions. We say that f and g share the value a CM if f and g have the same a-points with the same multiplicities (see [3]).

R. Nevanlinna [4], M. Ozawa [5], H. Ueda [6, 7], G. Brosch [8], E. Mues [9], H.X. Yi [10–13], P. Li [14], Q.C. Zhang [15] and other authors (see [2]) dealt with the problem of uniqueness of meromorphic functions that share three distinct values. Without loss of generality we may assume that 0, 1, ∞ are the shared values.

M. Ozawa [5] proved the following result.

Theorem A. *Let f and g be two distinct nonconstant entire functions of finite order such that f and g share 0 and 1 CM. If $\delta(0, f) > 1/2$, then $f \cdot g \equiv 1$.*

H. Ueda [6] removed the order restriction in Theorem A, and proved the following theorem.

Theorem B. *Let f and g be two distinct nonconstant meromorphic functions sharing 0, 1 and ∞ CM. If*

$$\limsup_{r\to\infty} \frac{N(r, f) + N(r, 0, f)}{T(r, f)} < \frac{1}{2},$$

then $f \cdot g \equiv 1$.

G. Brosch [8] proved the following result, which is an improvement of Theorem B.

Theorem C. *Let f and g be two distinct nonconstant meromorphic functions sharing 0, 1 and ∞ CM. If*

$$\limsup_{r\to\infty} \frac{\overline{N}(r, f) + \overline{N}(r, 0, f) - \frac{1}{2}m(r, 1, g)}{T(r, f)} < \frac{1}{2},$$

then $f \cdot g \equiv 1$.

In this paper, we prove the following theorem, which is improvement and extension of the above results.

Theorem 1. *Let f and g be two distinct nonconstant meromorphic functions sharing 0, 1 and ∞ CM. If there exists a set I of infinite linear measure such that*

(1)
$$\limsup_{\substack{r\to\infty \\ r\in I}} \frac{N_{1)}(r, f) + N_{1)}(r, 0, f) - m(r, 1, g)}{T(r, f)} < 1,$$

then f and g satisfy one of the following relations:

(i) $f = \dfrac{e^{s\gamma} - 1}{e^{-(k+1-s)\gamma} - 1}, \quad g = \dfrac{e^{-s\gamma} - 1}{e^{(k+1-s)\gamma} - 1},$

(ii) $f = \dfrac{e^{(k+1)\gamma} - 1}{e^{s\gamma} - 1}, \quad g = \dfrac{e^{-(k+1)\gamma} - 1}{e^{-s\gamma} - 1},$

(iii) $f = \dfrac{e^{s\gamma} - 1}{e^{(k+1)\gamma} - 1}, \quad g = \dfrac{e^{-s\gamma} - 1}{e^{-(k+1)\gamma} - 1},$

where s and k are positive integers such that $1 \leq s \leq k$, s and $k+1$ are relatively prime, γ is a nonconstant entire function, and

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$$(2)\qquad N_{1)}(r,f)+N_{1)}(r,0,f)-m(r,1,g) = \Big(1-\frac{1}{k}\Big)T(r,f)+S(r,f).$$

Example 1. Let

$$f(z)=\frac{e^z-1}{e^z+1},\quad g(z)=\frac{e^{-z}-1}{e^{-z}+1},$$

Example 2. Let

$$f(z)=\frac{e^z-1}{-e^{2z}-1},\quad g(z)=\frac{e^{-z}-1}{-e^{-2z}-1}.$$

Example 3. Let

$$f(z)=\frac{e^z-1}{-e^{-z}-1},\quad g(z)=\frac{e^{-z}-1}{-e^{z}-1}.$$

It is easy to show, from the above examples, that the assumption (1) in Theorem 1 is sharp.

As an immediate consequence of Theorem 1, we have

Corollary 1. *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If there exists a set I of infinite linear measure such that*

$$(3)\qquad \limsup_{\substack{r\to\infty\\ r\in I}}\frac{N_{1)}(r,f)+N_{1)}(r,0,f)-m(r,1,g)}{T(r,f)}<\frac{1}{2},$$

then f is a Möbius transformation of g,

$$N_{1)}(r,f)+N_{1)}(r,0,f)-m(r,1,g)=S(r,f),$$

and f and g satisfy one of the following relations:
(i) $f\,g\equiv 1$; (ii) $(f-1)\,(g-1)\equiv 1$;
(iii) $f+g\equiv 1$.

2. Some lemmas. The following lemmas will be needed in the proof of our theorems.

Lemma 1 ([12, Lemma 4]). *Let f and g be two nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If $f\not\equiv g$, then*

$$N_{(2}(r,f)+N_{(2}(r,0,f)+N_{(2}(r,1,f)=S(r,f).$$

Lemma 2 ([15, Lemma 7]). *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If f is a Möbius transformation of g, then f and g satisfy one of the following relations*:
(i) $f\,g\equiv 1$, (ii) $(f-1)(g-1)\equiv 1$,
(iii) $f+g\equiv 1$, (iv) $(f-c)\,(g+c-1)\equiv c(1-c)$,
(v) $f\equiv cg$, (vi) $f+(c-1)g\equiv c$,
where $c\,(\neq 0,1)$ is a constant.

Let f and g be two distinct nonconstant meromorphic functions sharing 0, 1 and ∞ CM. We use $N_0(r)$ to denote the counting function of the zeros of $f-g$ that are not zeros of f, $f-1$ and $1/f$.

The following lemma is essentially due to Q.C. Zhang [15].

Lemma 3 (see [15, Proof of Theorem 1 and Theorem 2]). *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM, and let $N_0(r)\neq S(r,f)$. If f is a Möbius transformation of g, then*

$$N_0(r)=T(r,f)+S(r,f).$$

If f is not any Möbius transformation of g, then

$$N_0(r)\le\frac{1}{2}T(r,f)+S(r,f),$$

and f and g assume one of the following relations:

$$\text{(i)}\quad f\equiv\frac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1},\qquad g\equiv\frac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1};$$

$$\text{(ii)}\quad f\equiv\frac{e^{(k+1)\gamma}-1}{e^{s\gamma}-1},\qquad g\equiv\frac{e^{-(k+1)\gamma}-1}{e^{-s\gamma}-1};$$

$$\text{(iii)}\quad f\equiv\frac{e^{s\gamma}-1}{e^{(k+1)\gamma}-1},\qquad g\equiv\frac{e^{-s\gamma}-1}{e^{-(k+1)\gamma}-1};$$

where γ is a nonconstant entire function, s and $k\,(\ge 2)$ are positive integers such that s and $k+1$ are relatively prime and $1\le s\le k$.

Lemma 4 (see [2, Theorem 5.13], [9, Theorem 5] or [12, (18)]). *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If f is not any Möbius transformation of g, then*

$$(4)\qquad T(r,f)+T(r,g)=N(r,f)+N(r,0,f) +N(r,1,f)+N_0(r)+S(r,f).$$

3. Proof of Theorem 1 and Corollary 1.

3.1. Proof of Theorem 1. Since $f\not\equiv g$, by Lemma 1, it follows that for $a=0,1,\infty$

$$(5)\qquad N(r,a,f)=N_{1)}(r,a,f)+S(r,f).$$

We discuss the following three cases.

Case 1. Suppose that f is a Möbius transformation of g. By Lemma 2, we know that f and g satisfy one of the six relations in Lemma 2. We distinguish the following six subcases.

Subcase 1.1. Assume that f and g satisfy the relation (i) in Lemma 2. Then 0 and ∞ are Picard exceptional values of f and g. Thus, we may assume that $f=-e^{\gamma}$ and $g=-e^{-\gamma}$, where γ is a nonconstant entire function. From this we obtain the relation (i) in Theorem 1 with $k = 1$.

Subcase 1.2. Assume that f and g satisfy the relation (ii) in Lemma 2. Then 1 and ∞ are Picard exceptional values of f and g. In the same manner as Subcase 1.1, we can obtain the relation (ii) in Theorem 1 with $k = 1$.

Subcase 1.3. Assume that f and g satisfy the relation (iii) in Lemma 2. Then 0 and 1 are Picard exceptional values of f and g. In the same manner as Subcase 1.1, we can obtain the relation (iii) in Theorem 1 with $k = 1$.

Subcase 1.4. Assume that f and g satisfy the relation (iv) in Lemma 2. Then c and ∞ are Picard exceptional values of f, $1-c$ and ∞ are Picard exceptional values of g. Thus, we may assume that $f = c(e^{\gamma}+1)$ and $g = (1-c)(e^{-\gamma}+1)$, where γ is a nonconstant entire function. From this we obtain

$$N_{1)}(r,f)+N_{1)}(r,0,f)-m(r,1,g) = T(r,f)+S(r,f),$$

which contradicts the assumption (1) in Theorem 1.

Subcase 1.5. Assume that f and g satisfy the relation (v) in Lemma 2. Then 1 and c are Picard exceptional values of f, 1 and $1/c$ are Picard exceptional values of g. In the same manner as Subcase 1.4, we have a contradiction.

Subcase 1.6. Assume that f and g satisfy the relation (vi) in Lemma 2. Then 0 and c are Picard exceptional values of f, 0 and $c/(c-1)$ are Picard exceptional values of g. In the same manner as Subcase 1.4, we also have a contradiction.

Case 2. Suppose that $N_0(r) \neq S(r,f)$, and that f is not any Möbius transformation of g. By Lemma 3, we know that f and g satisfy one of the three relations in Lemma 3. We distinguish the following three subcases.

Subcase 2.1. Assume that f and g satisfy the relation (i) in Lemma 3. From this we obtain the relation (i) in Theorem 1 with $k \geq 2$. Since $k\,(\geq 2)$ and s are positive integers such that $1 \leq s \leq k$, s and $k+1$ are relatively prime, we have

$$\begin{aligned} T(r,f) &= k\,T(r,e^{\gamma}) + S(r,f),\\ N_{1)}(r,f) &= (k-s)\,T(r,e^{\gamma}) + S(r,f),\\ N_{1)}(r,0,f) &= (s-1)\,T(r,e^{\gamma}) + S(r,f),\\ m(r,1,g) &= S(r,f). \end{aligned}$$

From the above we get (2).

Subcase 2.2. Assume that f and g satisfy the relation (ii) in Lemma 3. From this we obtain the relation (ii) in Theorem 1 with $k \geq 2$. In the same manner as Subcase 2.1, we have

$$\begin{aligned} T(r,f) &= k\,T(r,e^{\gamma}) + S(r,f),\\ N_{1)}(r,f) &= (s-1)\,T(r,e^{\gamma}) + S(r,f),\\ N_{1)}(r,0,f) &= k\,T(r,e^{\gamma}) + S(r,f),\\ m(r,1,g) &= s\,T(r,e^{\gamma}) + S(r,f). \end{aligned}$$

From this we get (2).

Subcase 2.3. Assume that f and g satisfy the relation (iii) in Lemma 3. From this we obtain the relation (iii) in Theorem 1 with $k \geq 2$. In the same manner as Subcase 2.1, we have

$$\begin{aligned} T(r,f) &= k\,T(r,e^{\gamma}) + S(r,f),\\ N_{1)}(r,f) &= k\,T(r,e^{\gamma}) + S(r,f),\\ N_{1)}(r,0,f) &= (s-1)\,T(r,e^{\gamma}) + S(r,f),\\ m(r,1,g) &= s\,T(r,e^{\gamma}) + S(r,f). \end{aligned}$$

From this we also get (2).

Case 3. Suppose that $N_0(r) = S(r,f)$, and that f is not any Möbius transformation of g. By Lemma 4, we can obtain (4). Obviously,

$$\begin{aligned} (6)\qquad m(r,1,g) &= T(r,g) - N(r,1,g) + O(1)\\ &= T(r,g) - N(r,1,f) + O(1). \end{aligned}$$

From (4), (5) and (6) we get

$$N_{1)}(r,f)+N_{1)}(r,0,f)-m(r,1,g) = T(r,f)+S(r,f),$$

which contradicts the assumption (1) in Theorem 1.

Theorem 1 is thus completely proved.

3.2. Proof of Corollary 1. By Theorem 1, from (3) we know that f and g satisfy one of the three relations in Theorem 1 with $k = s = 1$. From this we get that f and g satisfy one of the three relations in Corollary 1.

4. Concluding remarks. By the proof of Theorem 1, we have the following theorem, which is a supplement of Theorem 1.

Theorem 2. *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM. Then there exists a set E of finite linear measure such that*

$$\limsup_{\substack{r\to\infty\\ r\notin E}} \frac{N_{1)}(r,f)+N_{1)}(r,0,f)-m(r,1,g)}{T(r,f)} \leq 1.$$

Obviously,

$$\begin{aligned} &N_{1)}(r,f)+N_{1)}(r,0,f)-m(r,1,g)\\ &\qquad \leq N_{1)}(r,f)+N_{1)}(r,0,f)-\frac{1}{2}\,m(r,1,g) \end{aligned}$$

$$\le N_{1)}(r,f)+N_{1)}(r,0,f).$$

From the above and proceeding as in the proof of Theorem 1, we can prove the following results.

Theorem 3. *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If there exists a set I of infinite linear measure such that*

$$\limsup_{\substack{r\to\infty\\ r\in I}}\frac{N_{1)}(r,f)+N_{1)}(r,0,f)-\frac{1}{2}\,m(r,1,g)}{T(r,f)}<1,$$

then f and g satisfy one of the following relations:

(i) $f=\dfrac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1},\quad g=\dfrac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1},$

(ii) $f=e^{k\gamma}+\cdots+e^{\gamma}+1,$
$g=e^{-k\gamma}+\cdots+e^{-\gamma}+1,$

(iii) $f=\dfrac{e^{\gamma}-1}{e^{(k+1)\gamma}-1},\quad g=\dfrac{e^{-\gamma}-1}{e^{-(k+1)\gamma}-1},$

where k and s are positive integers such that $1\le s\le k$, s and $k+1$ are relatively prime, and γ is a nonconstant entire function, and

$$N_{1)}(r,f)+N_{1)}(r,0,f)-\frac{1}{2}\,m(r,1,g)=\Big(1-\frac{1}{k}\Big)T(r,f)+S(r,f),$$

in (i), *and*

$$N_{1)}(r,f)+N_{1)}(r,0,f)-\frac{1}{2}\,m(r,1,g)=\Big(1-\frac{1}{2k}\Big)T(r,f)+S(r,f),$$

in (ii) *and* (iii).

It is easy to see that Theorem 3 is improvement and extension of Theorem C. By Theorem 3, we have the following Corollary.

Corollary 2. *Let f and g be two distinct meromorphic functions sharing* 0, 1 *and* ∞ *CM. If f is not any Möbius transformation of g, and there exists a set I of infinite linear measure such that*

$$\limsup_{\substack{r\to\infty\\ r\in I}}\frac{N_{1)}(r,f)+N_{1)}(r,0,f)-\frac{1}{2}\,m(r,1,g)}{T(r,f)}\le\frac{1}{2},$$

then

$$N_{1)}(r,f)+N_{1)}(r,0,f)-\frac{1}{2}\,m(r,1,g)=\frac{1}{2}\,T(r,f)+S(r,f),$$

and f and g satisfy the following relation:

$$f=\frac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1},\quad g=\frac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1},$$

where $k=2$ and $s=1$ or 2, γ is a nonconstant entire function.

Theorem 4. *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If there exists a set I of infinite linear measure such that*

$$\limsup_{\substack{r\to\infty\\ r\in I}}\frac{N_{1)}(r,f)+N_{1)}(r,0,f)}{T(r,f)}<1, \tag{7}$$

then f and g satisfy the following relation:

$$f=\frac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1},\quad g=\frac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1},$$

where s and k are positive integers such that $1\le s\le k$, s and $k+1$ are relatively prime, γ is a nonconstant entire function, and

$$N_{1)}(r,f)+N_{1)}(r,0,f)=\Big(1-\frac{1}{k}\Big)T(r,f)+S(r,f).$$

It is easy to see that Theorem 4 is improvement and extension of Theorem A and Theorem B.

Example 4. Let

$$f(z)=\frac{2}{e^{z}+1},\quad g(z)=\frac{2}{e^{-z}+1}.$$

Example 5. Let

$$f(z)=\frac{1}{2}\,(e^{z}+1),\quad g(z)=\frac{1}{2}\,(e^{-z}+1).$$

Example 6. Let

$$f(z)=\frac{e^{z^2}-1}{e^{z}-1},\quad g(z)=\frac{e^{-z^2}-1}{e^{-z}-1}.$$

It is easy to show, from the above examples, that the assumption (7) in Theorem 4 is sharp.

By Theorem 4, we have the following:

Corollary 3. *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If there exists a set I of infinite linear measure such that*

$$\frac{1}{2}\le\limsup_{\substack{r\to\infty\\ r\in I}}\frac{N_{1)}(r,f)+N_{1)}(r,0,f)}{T(r,f)}<\frac{2}{3},$$

then f and g satisfy one of the following relations:

(i) $f=\dfrac{e^{\gamma}-1}{e^{-2\gamma}-1},\quad g=\dfrac{e^{-\gamma}-1}{e^{2\gamma}-1},$

(ii) $f=\dfrac{e^{2\gamma}-1}{e^{-\gamma}-1},\quad g=\dfrac{e^{-2\gamma}-1}{e^{\gamma}-1},$

where γ is a nonconstant entire function, and

$$N_{1)}(r,f)+N_{1)}(r,0,f)=\frac{1}{2}\,T(r,f)+S(r,f).$$

Corollary 4. *Let f and g be two distinct nonconstant entire functions sharing* 0 *and* 1 *CM. If* $\delta(0, f) > 0$*, then f and g satisfy the following relation*:

$$f = -e^{k\gamma} - \cdots - e^{\gamma}, \quad g = -e^{-k\gamma} - \cdots - e^{-\gamma},$$

where k is a positive integer, γ is a nonconstant entire function, and

$$N_{1)}(r, 0, f) = \Big(1 - \frac{1}{k}\Big)T(r, f) + S(r, f).$$

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