Univalency of certain analytic functions

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Abstract: Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in the open unit disk $\mathbf{U}$ with $f(0) = 0$ and $f'(0) = 1$. Using $g(z) \in \mathcal{A}$, the subclass $\mathcal{T}(\lambda, \mu, g)$ of $\mathcal{A}$ consisting of functions $f(z)$ is introduced. The object of the present paper is to consider some univalence conditions for functions $f(z)$ belonging to the class $\mathcal{T}(\lambda, \mu, g)$ applying the subordination properties of analytic functions.

Key words: Analytic function; univalent function; starlike function; subordination.

1. Introduction. Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\mathbf{U} = \{z \in \mathbf{C} : |z| < 1\}$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbf{U}$. Let $g(z) \in \mathcal{A}$ with $(g(z)/z) \neq 0$ for $z \in \mathbf{U}$. Then we say that $f(z) \in \mathcal{A}$ is in the class $\mathcal{T}(\lambda, \mu, g)$ if and only if it satisfies the conditions $(f(z)/z) \neq 0$ in $\mathbf{U}$ and

$$\left| z^2 \left(\frac{f'(z)}{f(z)^2} - \frac{g'(z)}{g(z)^2} \right) - \lambda z^2 \left(\frac{z}{f(z)} - \frac{z}{g(z)} \right)'' \right| < \mu \quad (z \in \mathbf{U}), \tag{1}$$

where λ is complex with $\mathrm{Re}(\lambda) \geqq 0$ and $\mu > 0$.

Let $f(z)$ and $g(z)$ be analytic in $\mathbf{U}$. Then $f(z)$ is said to be subordinate to $g(z)$ in $\mathbf{U}$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbf{U}$ such that $|w(z)| \leqq |z|$ and $f(z) = g(w(z))$ for $z \in \mathbf{U}$. If $g(z)$ is univalent in $\mathbf{U}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbf{U}) \subset g(\mathbf{U})$.

To discuss our problems, we need to recall here the following lemmas.

Lemma 1.1. *Let* $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ $(n \in \mathbf{N} = 1, 2, 3, \cdots)$ *be analytic in* $\mathbf{U}$ *and let* $h(z)$ *be analytic and convex univalent in* $\mathbf{U}$ *with* $h(0) = 1$. *If*

$$p(z) + \frac{1}{c} z p'(z) \prec h(z)$$

for $\mathrm{Re}(c) \geqq 0$ *and* $c \neq 0$, *then*

$$p(z) \prec \frac{c}{n} z^{-(c/n)} \int_0^z t^{(c/n)-1} h(t)\, dt.$$

The above lemma is due to Miller and Mocanu ([2], p. 170).

Lemma 1.2. *Let* $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ $(n \in \mathbf{N})$ *be analytic in* $\mathbf{U}$ *and* $h(z)$ *be analytic and starlike* (*with respect to the origin*) *univalent in* $\mathbf{U}$ *with* $h(0) = 0$. *If* $zp'(z) \prec h(z)$, *then*

$$p(z) \prec 1 + \frac{1}{n} \int_0^z \frac{h(t)}{t}\, dt.$$

Lemma 1.2 was given by Yang [5].

2. Univalency of functions. Now, our first result is contained in

Theorem 2.1. *If* $f(z) \in \mathcal{T}(\lambda, \mu, g)$ *and*

$$\delta(g) \geqq \frac{\mu}{|1 + 2\lambda|} \tag{2}$$

for $\mathrm{Re}(\lambda) \geqq 0$, $\mu > 0$ *and*

$$\delta(g) = \inf \left\{ \left| \frac{\frac{1}{g(z_1)} - \frac{1}{g(z_2)}}{z_1 - z_2} \right| : z_1 \neq z_2, \; 0 < |z_1| < 1,\, 0 < |z_2| < 1 \right\}, \tag{3}$$

then $f(z) \in \mathcal{S}$.

Proof. Let us define the function $p(z)$ by

$$p(z) = 1 + z^2 \left(\frac{f'(z)}{f(z)^2} - \frac{g'(z)}{g(z)^2} \right) \tag{4}$$

for $f(z) \in \mathcal{T}(\lambda, \mu, g)$. Then $p(z) = 1 + p_2 z^2 + p_3 z^3 + \cdots$ is analytic in $\mathbf{U}$,

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$$p(z) = 1 + \left(\frac{z}{f(z)} - \frac{z}{g(z)}\right) - z\left(\frac{z}{f(z)} - \frac{z}{g(z)}\right)'$$

and

$$(5) \qquad zp'(z) = -z^2\left(\frac{z}{f(z)} - \frac{z}{g(z)}\right)''.$$

Hence

$$p(z) + \lambda z p'(z) = 1 + z^2\left(\frac{f'(z)}{f(z)^2} - \frac{g'(z)}{g(z)^2}\right) - \lambda z^2\left(\frac{z}{f(z)} - \frac{z}{g(z)}\right)''$$

and it follows from (1) that

$$p(z) + \lambda z p'(z) \prec 1 + \mu z.$$

Since $h(z) = 1 + \mu z$ is analytic and convex univalent in $\mathbf{U}$ with $h(0) = 1$, an application of Lemma 1.1 with $n = 2$ and $c = (1/\lambda)$ yields

$$(6) \qquad p(z) \prec 1 + \frac{\mu}{1+2\lambda} z,$$

where $\mathrm{Re}(\lambda) \geqq 0$, $\lambda \neq 0$, and $\mu > 0$. It is clear that the subordination (6) is also valid for $\lambda = 0$.

From (4), (6) and the Schwarz lemma, we have that

$$(7) \qquad \left|\frac{f'(z)}{f(z)^2} - \frac{g'(z)}{g(z)^2}\right| \leqq \frac{\mu}{|1+2\lambda|} \quad (z \in \mathbf{U})$$

for $\mathrm{Re}(\lambda) \geqq 0$ and $\mu > 0$. Since

$$\int_{z_1}^{z_2} \left(\frac{f'(t)}{f(t)^2} - \frac{g'(t)}{g(t)^2}\right) dt = \left(\frac{1}{f(z_1)} - \frac{1}{g(z_1)}\right) - \left(\frac{1}{f(z_2)} - \frac{1}{g(z_2)}\right),$$

where $z_1 \in \mathbf{U}, z_2 \in \mathbf{U}, z_1 \neq z_2$, and the path of the integration is the line segment from z_1 to z_2, it follows from (7) that

$$(8) \qquad \left|\left(\frac{1}{f(z_1)} - \frac{1}{f(z_2)}\right) - \left(\frac{1}{g(z_1)} - \frac{1}{g(z_2)}\right)\right| \leqq \frac{\mu}{|1+2\lambda|}|z_1 - z_2|.$$

We wish to show that $f(z_1) \neq f(z_2)$. If we suppose that $f(z_1) = f(z_2)$, then (8) becomes

$$\left|\frac{1}{g(z_1)} - \frac{1}{g(z_2)}\right| \leqq \frac{\mu}{|1+2\lambda|}|z_1 - z_2|,$$

where $z_1 \neq z_2$ and $z_1 z_2 \neq 0$. This contradicts the conditions (2) and (3) of the theorem. Hence we conclude that $f(z) \in \mathcal{S}$. □

Corollary 2.1. *Let* $f(z) \in \mathcal{A}$ *satisfy* $(f(z)/z) \neq 0$ *in* $\mathbf{U}$ *and*

$$\left|\frac{z^2 f'(z)}{f(z)^2} - \frac{1+2\alpha z}{(1+\alpha z)^2} - \lambda z^2 \left\{\left(\frac{z}{f(z)}\right)'' - \frac{2\alpha^2}{(1+\alpha z)^3}\right\}\right| < \mu \quad (z \in \mathbf{U}),$$

where $\mathrm{Re}(\lambda) \geqq 0$, $\mu > 0$, $|\alpha| < (1/2)$ *and*

$$\frac{\mu}{|1+2\lambda|} \leqq \frac{1-2|\alpha|}{(1-|\alpha|)^2}.$$

Then $f(z) \in \mathcal{S}$.

Proof. Let $g(z) = z + \alpha z^2$ with $|\alpha| < (1/2)$. Then, for $z_1 \neq z_2$, $0 < |z_1| < 1$ and $0 < |z_2| < 1$,

$$\begin{aligned}\left|\frac{z_1 - z_2}{\frac{1}{g(z_1)} - \frac{1}{g(z_2)}}\right| &= \left|\frac{z_1 z_2 (1+\alpha z_1)(1+\alpha z_2)}{1+\alpha(z_1+z_2)}\right| \\ &= |z_1 z_2| \left|1 + \frac{\alpha^2 z_1 z_2}{1+\alpha(z_1+z_2)}\right| \\ &\leqq |z_1 z_2| \left(1 + \frac{|\alpha|^2 |z_1 z_2|}{1-|\alpha|(|z_1|+|z_2|)}\right) \\ &< \frac{(1-|\alpha|)^2}{1-2|\alpha|}.\end{aligned}$$

Thus we easily have

$$\delta(g) = \frac{1-2|\alpha|}{(1-|\alpha|)^2} > 0.$$

Now the corollary follows immediately from Theorem 2.1. □

Remark 1. Taking $\lambda = \alpha = 0$ and $\mu = 1$, the corollary reduces to the result by Ozaki and Nunokawa [4].

Corollary 2.2. *Let*

$$(9) \qquad f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} \alpha_n z^n} \in \mathcal{A} \quad \textit{and}$$

$$g(z) = \frac{z}{1 + \sum_{n=1}^{\infty} \beta_n z^n} \in \mathcal{A},$$

and let $\mathrm{Re}(\lambda) \geqq 0$, $\mu > 0$, $\delta(g) \geqq (\mu/|1+2\lambda|)$, *where* $\delta(g)$ *is given by* (3). *If*

$$(10) \qquad \sum_{n=2}^{\infty} (n-1)|1+n\lambda||\alpha_n - \beta_n| \leqq \mu,$$

then $f(z) \in \mathcal{S}$.

Proof. From (9) and (10), we have

$$\left|z^2\left(\frac{f'(z)}{f(z)^2} - \frac{g'(z)}{g(z)^2}\right) - \lambda z^2 \left(\frac{z}{f(z)} - \frac{z}{g(z)}\right)''\right| = \left|-\sum_{n=2}^{\infty}(n-1)(1+n\lambda)(\alpha_n - \beta_n) z^n\right|$$

$$\leqq \sum_{n=2}^{\infty}(n-1)|1+n\lambda||\alpha_n-\beta_n| \leqq \mu$$

for $z \in \mathbf{U}$. Hence $f(z) \in \mathcal{T}(\lambda,\mu,g) \subset \mathcal{S}$ by using Theorem 1.1. □

Next, we derive

Theorem 2.2. *Let* $0 \leqq \lambda_1 < \lambda_2$ *and* $\mu > 0$. *Then* $\mathcal{T}(\lambda_2,\mu,g) \subset \mathcal{T}(\lambda_1,\mu,g)$.

Proof. Let the function $f(z)$ be in the class $\mathcal{T}(\lambda_2,\mu,g)$. Then

$$\left|z^2\left(\frac{f'(z)}{f(z)^2}-\frac{g'(z)}{g(z)^2}\right)-\lambda_2 z^2\left(\frac{z}{f(z)}-\frac{z}{g(z)}\right)''\right| < \mu \quad (z \in \mathbf{U})$$

and from (7) in the proof of Theorem 1.1, we have

$$\left|\frac{f'(z)}{f(z)^2}-\frac{g'(z)}{g(z)^2}\right| \leqq \frac{\mu}{1+2\lambda_2} < \mu \quad (z \in \mathbf{U}).$$

Therefore, for $0 \leqq \lambda_1 < \lambda_2$,

$$\left|z^2\left(\frac{f'(z)}{f(z)^2}-\frac{g'(z)}{g(z)^2}\right)-\lambda_1 z^2\left(\frac{z}{f(z)}-\frac{z}{g(z)}\right)''\right|$$

$$=\left|\frac{\lambda_1}{\lambda_2}\left\{z^2\left(\frac{f'(z)}{f(z)^2}-\frac{g'(z)}{g(z)^2}\right)-\lambda_2 z^2\left(\frac{z}{f(z)}-\frac{z}{g(z)}\right)''\right\} + \left(1-\frac{\lambda_1}{\lambda_2}\right)z^2\left(\frac{f'(z)}{f(z)^2}-\frac{g'(z)}{g(z)^2}\right)\right|$$

$$< \frac{\lambda_1}{\lambda_2}\mu + \left(1-\frac{\lambda_1}{\lambda_2}\right)\mu = \mu \qquad (z \in \mathbf{U}).$$

This shows that $f(z) \in \mathcal{T}(\lambda_1,\mu,g)$. □

Next, we find the radius of univalency for functions $f(z) \in \mathcal{T}(\lambda,\mu,g)$.

Theorem 2.3. *Let* $f(z) \in \mathcal{T}(\lambda,\mu,g)$ *with* $\mathrm{Re}(\lambda) \geqq 0$, $\mu > 0$ *and* $g(z) \in \mathcal{S}$. *Then* $f(z)$ *is univalent for*

$$|z| < \sqrt{\frac{|1+2\lambda|}{\mu+|1+2\lambda|}}.$$

Proof. To prove that $f(z)$ is univalent in $|z| \leqq \rho$ $(0 < \rho < 1)$, it suffices to show that $f(z)$ is univalent on $|z| = \rho$. Let $z_1 \neq z_2$ and $|z_1| = |z_2| = \rho$. Then from the proof of Theorem 2.1, we see that $f(z_1) = f(z_2)$ leads to

$$\left|\frac{1}{g(z_1)}-\frac{1}{g(z_2)}\right| \leqq \frac{\mu}{|1+2\lambda|}|z_1-z_2|. \tag{11}$$

On the other hand, since $g(z) \in \mathcal{S}$, it is known (see, e.g., Duren [1, p. 127]) that

$$\left|\frac{g(z_1)-g(z_2)}{z_1-z_2}\right| \geqq \frac{1-\rho^2}{\rho^2}|g(z_1)g(z_2)|,$$

and hence

$$\left|\frac{g(z_1)-g(z_2)}{z_1-z_2}\right| > \frac{\mu}{|1+2\lambda|}|g(z_1)g(z_2)| \tag{12}$$

for

$$0 < \rho < \rho^* = \sqrt{\frac{|1+2\lambda|}{\mu+|1+2\lambda|}}.$$

In view of (11) and (12), we know that $f(z_1) \neq f(z_2)$ and so $f(z)$ is univalent on $|z| = \rho$ $(0 < \rho < \rho^*)$. Thus we complete the proof of the theorem. □

For $\lambda = 0$ and $\mu > 0$, Theorem 2.3 yields the following corollary.

Corollary 2.3. *Let* $f(z) \in \mathcal{A}$ *with* $(f(z)/z) \neq 0$ *for* $z \in \mathbf{U}$ *and let* $g(z) \in \mathcal{S}$. *If* $f(z)$ *satisfies*

$$\left|\frac{f'(z)}{f(z)^2}-\frac{g'(z)}{g(z)^2}\right| \leqq \mu \quad (z \in \mathbf{U}),$$

then $f(z)$ *is univalent for*

$$|z| < \frac{1}{\sqrt{1+\mu}} \quad (\mu > 0).$$

Furthermore, we derive

Theorem 2.4. *Let* $f(z) \in \mathcal{A}$, $g(z) \in \mathcal{A}$ *with* $f(z)g(z) \neq 0$ *for* $0 < |z| < 1$ *and*

$$\delta(g) \geqq 1, \tag{13}$$

where $\delta(g)$ *is given by* (3). *If*

$$\left|\left(\frac{z}{f(z)}-\frac{z}{g(z)}\right)''\right| \leqq 2 \quad (z \in \mathbf{U}), \tag{14}$$

then $f(z) \in \mathcal{S}$.

Proof. From (5) in the proof of Theorem 2.1 and the condition (14), we see that

$$zp'(z) \prec 2z,$$

where

$$p(z) = 1 + z^2\left(\frac{f'(z)}{f(z)^2}-\frac{g'(z)}{g(z)^2}\right) = 1 + p_2 z^2 + \cdots$$

is analytic in $\mathbf{U}$. Hence, by Lemma 1.2 with $h(z) = 2z$ and $n = 2$, we have that

$$p(z) \prec 1 + \frac{1}{2}\int_0^z \frac{h(t)}{t}\,dt = 1 + z,$$

which is equivalent to

$$\left|z^2\left(\frac{f'(z)}{f(z)^2}-\frac{g'(z)}{g(z)^2}\right)\right| < 1 \quad (z \in \mathbf{U}). \tag{15}$$

Now, from (13) and (15) and Theorem 2.1 with $\lambda = 0$ and $\mu = 1$, we conclude that $f(z) \in \mathcal{S}$. □

In [3, Theorem], Nunokawa, Obradović and Owa showed that

Theorem A. *Let $f(z) \in \mathcal{A}$ with $(f(z)/z) \neq 0$ for $z \in \mathbf{U}$ and*

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leqq 1 \quad (z \in \mathbf{U}).$$

Then $f(z) \in \mathcal{S}$.

Letting $g(z) = z$ in Theorem 2.4, we have an improvement of Theorem A as follows:

Corollary 2.4. *Let $f(z) \in \mathcal{A}$ with $(f(z)/z) \neq 0$ for $z \in \mathbf{U}$ and*

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leqq 2 \quad (z \in \mathbf{U}). \tag{16}$$

Then $f(z) \in \mathcal{S}$.

Remark 2. Recently, Yang and Liu [6] obtained the corollary by using the another method. Further, the bound 2 in (16) is best possible as shown by

$$f(z) = \frac{z}{(1+z)^2}.$$

3. Coefficient inequality. The coefficient inequality for $f(z)$ and $g(z)$ when $f(z) \in \mathcal{T}(\lambda, \mu, g)$ is shown in

Theorem 3.1. *Let*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{T}(\lambda, \mu, g),$$

where $\mathrm{Re}(\lambda) \geqq 0$, $\mu > 0$, *and*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Then

$$\left|(a_2^2 - a_3) - (b_2^2 - b_3)\right| \leqq \frac{\mu}{|1+2\lambda|}. \tag{17}$$

Proof. Since

$$\left(\frac{1}{g(z)} - \frac{1}{f(z)}\right)\Big|_{z=0} = a_2 - b_2,$$

from (7) in the proof of Theorem 2.1, we deduce that

$$\begin{aligned} &\left|\frac{1}{g(z)} - \frac{1}{f(z)} - (a_2 - b_2)\right| \\ &= \left|\int_0^z \left(\frac{f'(z)}{f(z)^2} - \frac{g'(z)}{g(z)^2}\right) dt\right| \\ &\leqq \frac{\mu}{|1+2\lambda|}|z| \quad (z \in \mathbf{U}). \end{aligned} \tag{18}$$

Note that

$$\begin{aligned} &\frac{1}{f(z)} - \frac{1}{g(z)} + (a_2 - b_2) \\ &= ((a_2^2 - a_3) - (b_2^2 - b_3))z + \sum_{n=2}^{\infty} c_n z^n. \end{aligned} \tag{19}$$

It follows from (18) and (19) that

$$\begin{aligned} &|(a_2^2 - a_3) - (b_2^2 - b_3)|^2 r^2 + \sum_{n=2}^{\infty} |c_n|^2 r^{2n} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1}{f(re^{i\theta})} - \frac{1}{g(re^{i\theta})} + (a_2 - b_2)\right|^2 d\theta \\ &\leqq \left(\frac{\mu}{|1+2\lambda|}\right)^2 r^2 \quad (0 < r < 1), \end{aligned}$$

which yields the coefficient inequality (17).

Finally, for $0 < \mu \leqq |1+2\lambda|$, it is readily verified that the equality in (17) is attained, for example, by

$$\begin{aligned} f(z) &= \frac{z}{(1-\alpha z)^2} \\ &= z + 2\alpha z^2 + 3\alpha^2 z^3 + \cdots \in \mathcal{T}(\lambda, \mu, g), \end{aligned}$$

where

$$g(z) = \frac{z}{(1-\beta z)^2} = z + 2\beta z^2 + 3\beta^2 z^3 + \cdots,$$

$0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$, and $|\alpha^2 - \beta^2| = (\mu/|1+2\lambda|)$. □

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