Characterizations of space forms by circles on their geodesic spheres

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Abstract: In this paper we characterize space forms by observing the extrinsic shape of circles on their geodesic spheres.

Key words: Curves of order 2; plane curves; geodesics; circles; geodesic spheres; space forms.

1. Introduction. A smooth curve γ on a complete Riemannian manifold M parametrized by its arclength is called a *curve of order* 2 if it satisfies the following nonlinear differential equation:

(C)

$$\|\nabla_{\dot\gamma}\dot\gamma\|^2\left\{\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma+\|\nabla_{\dot\gamma}\dot\gamma\|^2\dot\gamma\right\}=\langle\nabla_{\dot\gamma}\dot\gamma,\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma\rangle\nabla_{\dot\gamma}\dot\gamma,$$

where $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ on M. Typical examples of curves of order 2 are circles and plane curves. We call a smooth curve γ parametrized by its arclength a *circle* if it satisfies $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma=-k^2\dot\gamma$ with some nonnegative constant k. This condition is equivalent to the condition that there exist a nonnegative constant k and a field of unit vectors Y along this curve which satisfy $\nabla_{\dot\gamma}\dot\gamma=kY$ and $\nabla_{\dot\gamma}Y=-k\dot\gamma$. The constant k is called the *curvature* of γ. As we see $k=\|\nabla_{\dot\gamma}\dot\gamma\|$, we find circles are curves of order 2. Also we see geodesics are treated as circles of null curvature. A smooth curve is said to be a *plane curve* if it is locally contained on some real 2-dimensional totally geodesic submanifold. On a space form $M(c)$, which is either one of a standard sphere, a Euclidean space and a hyperbolic space, circles are plane curves, but it is not true in general (see [AMU] for circles on a complex projective space). Thus the class of curves of oder 2 is a wide class. For more detail we study in Section 2.

The aim of this paper is to characterize space forms in terms of the extrinsic shape of geodesics and circles on geodesic spheres in these spaces. In a space form $M(c)$, every geodesic sphere is a totally umbilic but not totally geodesic hypersurface with parallel second fundamental form. This tells us that every circle on each geodesic sphere is a circle in the ambient manifold $M(c)$. Motivated by this fact, we here establish characterizations of space forms from the viewpoint of their geodesic spheres. In the preceding paper[AM], we characterize space forms by observing the extrinsic shape of geodesics on their geodesic spheres. Our results are extensions of this result.

2. Curves of order 2. We devote this section to study some fundamental properties of curves of order 2. A smooth curve $\gamma=\gamma(s)$ parametrized by its arclength s is called a *Frenet curve of order* 2 *in the wider sense* if there exist a smooth unit vector field Y along γ which is orthogonal to $\dot\gamma$ and a smooth function κ satisfying

(F)

$$\nabla_{\dot\gamma}\dot\gamma(s)=\kappa(s)Y(s)\quad\text{and}\quad\nabla_{\dot\gamma}Y(s)=-\kappa(s)\dot\gamma(s).$$

We shall call κ the curvature function. When we can take κ as a positive function, we call γ a *Frenet curve of proper order* 2. For a Frenet curve of proper order 2 the function κ and the orthonormal frame $\{\dot\gamma, Y\}$ are called its *curvature* and *Frenet frame*, respectively. Trivially a circle of positive curvature is a Frnet curve of proper order 2. It is also clear that every geodesic is a Frenet curve of order 2 in the wider sense with an arbitrary parallel unit vector field along this geodesic which is orthogonal to the tangent vector field. For consistency, we call a curve a *Frenet curve of order* 2 if it is either a geodesic or a Frenet curve of proper order 2.

Lemma 1. *If a curve γ of order 2 satisfies $\|\nabla_{\dot\gamma}\dot\gamma(s)\|>0$ for all s, then it is a Frenet curve of proper order 2, whose curvature and Frenet frame are $\kappa(s)=\|\nabla_{\dot\gamma}\dot\gamma(s)\|$ and $\{\dot\gamma, Y=\nabla_{\dot\gamma}\dot\gamma/\|\nabla_{\dot\gamma}\dot\gamma\|\}$,*

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respectively.

Proof. If we put $\kappa(s) = \|\nabla_{\dot\gamma}\dot\gamma(s)\|$, we have $\kappa\kappa' = \langle\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma\rangle$. Therefore by the equation (C) the vector field $Y = (1/\kappa)\nabla_{\dot\gamma}\dot\gamma$ satisfies

$$\nabla_{\dot\gamma}Y = \frac{1}{\kappa^3}\left(\kappa^2\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma - \kappa\kappa'\nabla_{\dot\gamma}\dot\gamma\right) = -\kappa\dot\gamma,$$

which leads us to the conclusion. □

Following this lemma, for a curve γ of order 2 we shall call the nonnegative function $\|\nabla_{\dot\gamma}\dot\gamma\|$ its curvature. It should note that we admit a curve γ of order 2 to have points where $\nabla_{\dot\gamma}\dot\gamma$ vanishes. For a curve of order 2 we call such a point an *inflection point*, and call an *ordinary* point if it is not an inflection point. A cubic curve $y = x^3$ on a Euclidean xy-plane is a good example of Frenet curve of order 2 in the wider sense. One can easily find that the origin is an inflection point. We have to take care of treating inflection points.

Lemma 2. (1) *Every Frenet curve of order* 2 *in the wider sense is a curve of order* 2.

(2) *Every smooth plane curve parametrized by its arclength is also a curve of order* 2.

Proof. (1) By definition we find a Frenet curve γ of order 2 in the wider sense satisfies the following:

$$\begin{aligned}\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma(s) &= \nabla_{\dot\gamma}\big(\kappa(s)Y(s)\big)\\ &= -\kappa'(s)Y(s) - \kappa^2(s)\dot\gamma(s).\end{aligned}$$

This guarantees that γ satisfies eq. (C).

(2) Let γ be a smooth plane curve parametrized by its arclength. For each s_0 we have a real 2-dimensional totally geodesic submanifold S and positive number δ such that the restriction $\gamma|_{(s_0-\delta,s_0+\delta)}$ lies on S. We then have a local smooth unit vector field V along $\gamma|_{(s_0-\delta,s_0+\delta)}$ which is orthogonal to $\dot\gamma$ and is tangent to S. Since $\|\dot\gamma\| = \|V\| = 1$, we see $\langle\dot\gamma, V\rangle = \langle V, \nabla_{\dot\gamma}V\rangle = 0$. Since S is 2-dimensional and totally geodesic, we see $\nabla_{\dot\gamma}\dot\gamma$ is proportional to V and $\nabla_{\dot\gamma}V$ is proportional to $\dot\gamma$. By differentiating $\langle\dot\gamma, V\rangle = 0$, we find $\nabla_{\dot\gamma}\dot\gamma = \nu V$ and $\nabla_{\dot\gamma}V = -\nu\dot\gamma$ with a function ν. As the argument in (1) stands locally we obtain the conclusion. □

When γ is a Frenet curve of order 2 in the wider sense and is not a geodesic, since Y is parallel on each interval consisting of inflection points,the pair (κ, Y) is determined up to their signatures, that is either (κ, Y) or $(-\kappa, -Y)$ satisfies eq. (F). In a Euclidean space, each Frenet curve of order 2 in the wider sense lies on some single plane. However, in general a plane curve is not necessarily a Frenet curve of order 2 in the wider sense. On the other hand, on a complex projective space we have circles which are not plane curves. Hence a Frenet curve of order 2 in the wider sense is not necessarily a plane curve. Thus the notion of curves of order 2 is an extention of both of the notion of plane curves and that of Frenet curves of order 2. We here give interesting examples.

Example 1. Let γ_1 be a smooth curve in a Euclidean space $\mathbf{R}^3$ defined by

$$\gamma_1(t) = \begin{cases}(t, e^{-1/t^2}, 0), & t < 0,\\ (0,0,0), & t = 0,\\ (t, 0, e^{-1/t^2}), & t > 0.\end{cases}$$

When we reparametrize t to the arclength parameter s, the curve $\gamma(s)$ satisfies eq. (C). This curve is niether a plane curve nor a Frenet curve of order 2 in the wider sense. We can not take a plane which contains γ_1 locally at the origin. Also the origin is an isolated inflection point and we see that we can not smoothly extend the vector field $\nabla_{\dot\gamma}\dot\gamma(s)/\|\nabla_{\dot\gamma}\dot\gamma(s)\|$ $(-\epsilon < s < 0,\ 0 < s < \epsilon)$ along γ to the origin.

Example 2. We should note that a similar curve in $\mathbf{R}^3$ defined by reparametrizing the curve

$$\gamma_2(t) = \begin{cases}(t, e^{-1/(t+1)^2}, 0), & t < -1,\\ (t,0,0), & -1 \le t \le 0,\\ (t, 0, e^{-1/t^2}), & t > 0,\end{cases}$$

is a plane curve which is not a Frenet curve of order 2 in the wider sense.

At an inflection point we should take care in handling curves of order 2. For example, the differential eq. (C) may have a bifurcation point.

Example 3. Let γ_3 be a smooth curve in a Euclidean space $\mathbf{R}^3$ defined by

$$\gamma_3(t) = \begin{cases}(t, e^{-1/t^2}, 0), & t \ne 0,\\ (0,0,0), & t = 0.\end{cases}$$

When we reparametrize t to the arclength parameter s, the curve $\gamma_3(s)$ also satisfies the eq. (C). Comparing this with $\gamma_1(s)$ in Example 1, we find a solution of (C) branches at the origin. We remark that the origin is an isolated inflection point of these curves. We also remark that the curve γ_3 is a plane curve and also a Frenet curve of order 2 in the wider sense.

On the contrary, we have the following result on Frenet curves of order 2 in the wider sense in a complete Riemannian manifold M. Let $\kappa(s), -\infty <$

$s < \infty$ be a smooth function. Given a pair $X, Y \in T_xM$ of orthonormal vectors at an arbitrary point $x \in M$, we have a unique Frenet curve γ of order 2 in the wider sense with curvature function κ and with initial condition that $\gamma(0) = x, \dot\gamma(0) = X, \nabla_{\dot\gamma}\dot\gamma(0) = \kappa(0)Y$.

3. Characterizations of space forms. Let $G_p(r)$ be a geodesic sphere with center $p \in M$ and radius r. First we shall characterize a space form $M^n(c)$ of curvature c by observing the extrinsic shape of *geodesics* on a geodesic sphere $G_p(r)$ with sufficiently small radius r in $M^n(c)$.

Theorem 1. *Let M be a complete Riemannian manifold of dimension greater than 2. Then the following conditions are equivalent*:

(1) *M is a space form.*
(2) *For each point $p \in M$ there is a positive number ϵ_p such that every geodesic on a geodesic sphere $G_p(r)$ of M is a circle of positive curvature in M for each r with $0 < r < \epsilon_p$.*
(3) *For each point $p \in M$ there is a positive number ϵ_p such that every geodesic on a geodesic sphere $G_p(r)$ of M is a plane curve in M for each r with $0 < r < \epsilon_p$.*
(4) *For each point $p \in M$ there is a positive number ϵ_p such that every geodesic on a geodesic sphere $G_p(r)$ of M is a Frenet curve of order 2 in the wider sense in M for each r with $0 < r < \epsilon_p$.*
(5) *For each point $p \in M$ there is a positive number ϵ_p such that every geodesic on a geodesic sphere $G_p(r)$ of M is a curve of order 2 in M for each r with $0 < r < \epsilon_p$.*

Proof. As was mentioned in Introduction, in a space form $M(c)$ every geodesic sphere $G_p(r)$ with radius smaller than the injective radius is totally umbilic. Hence geodesics on $G_p(r)$ are circles in $M(c)$. Since circles on a space form are plane curves, in view of Section 2, what we have to show is that (5) implies (1).

Given an arbitrary point $p \in M$ we consider a geodesic sphere $G_p(r)$ of sufficiently small radius. For a unit tangent vector $v \in TG_p(r)$ we shall verify one of the following conditions holds:

i) $\langle Av, v\rangle = 0$,
ii) $Av = \lambda_v v$ for some $\lambda_v \in \mathbf{R}$,

where $\langle\ ,\ \rangle$ is the Riemannian metric on $G_p(r)$ and A is the shape operator of $G_p(r)$ in M. In order to see this we suppose $\langle Av, v\rangle \neq 0$. Let $\gamma = \gamma(s), -\delta < s < \delta$, be a geodesic segment on $G_p(r)$ with initial condition that $\gamma(0) = x, \dot\gamma(0) = v$ and with $\langle A\dot\gamma(s), \dot\gamma(s)\rangle \neq 0$ for all $s \in (-\delta, \delta)$. We denote by ∇ and $\widetilde\nabla$ the Riemannian connections of $G_p(r)$ and M, respectively. By the Gauss and the Weingarten formulae we have

$$\widetilde\nabla_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + \langle A\dot\gamma, \dot\gamma\rangle N = \langle A\dot\gamma, \dot\gamma\rangle N,$$
$$\widetilde\nabla_{\dot\gamma}N = -A\dot\gamma,$$

where N is a unit normal vector field of $G_p(r)$ in M. By hypothesis the curve γ is a curve of order 2 in the ambient space M. As $\|\widetilde\nabla_{\dot\gamma}\dot\gamma\| = \sqrt{\langle A\dot\gamma, \dot\gamma\rangle} > 0$, we find γ is a Frenet curve of proper order 2 in M by Lemma 1. Therefore we get $A\dot\gamma(s)$ is proportional to $\dot\gamma(s)$ for each s, in particular, the vector v at the point x is the principal curvature vector of $G_p(r)$ in M.

We now show that $G_p(r)$ is totally umbilic in the ambient space M. Choose an arbitrary point $x \in G_p(r)$ and take an orthonormal basis $\{v_1, \ldots, v_{n-1}\}$ of the tangent space $T_xG_p(r)$ as principal curvature vectors of $G_p(r)$ in M, that is, they satisfy $Av_i = \lambda_i v_i$ for $1 \leqq i \leqq n-1$. For distinct i, j ($1 \leqq i, j \leqq n-1$) the above discussion guarantees that one of the following conditions holds:

i) $\langle A(v_i + v_j), v_i + v_j\rangle = 0$,
ii) $A(v_i + v_j) = \lambda_{ij}(v_i + v_j)$ for some $\lambda_{ij} \in \mathbf{R}$.

In the second case, we take the inner products of both sides of the equality and vectors v_i, v_j. We then have $\lambda_i = \lambda_{ij} = \lambda_j$. In the first case, the equality leads us to $\lambda_j = -\lambda_i$. In order to see $\lambda_i = \lambda_j$ we consider the vector $v_i + 2v_j$. The previous discussion also guarantees that either $\langle A(v_i + 2v_j), v_i + 2v_j\rangle = 0$ or $A(v_i+2v_j) = \lambda(v_i+2v_j)$ holds for some $\lambda \in \mathbf{R}$. In the former case, the equality leads us to $\lambda_i + 4\lambda_j = 0$. In the latter case, by taking the inner products of both side of the equality and vectors v_i, v_j, we have $\lambda_i = \lambda = \lambda_j$. Hence we find in the first case that $\lambda_i = \lambda_j = 0$. Thus we see that $\lambda_i = \lambda_j$ for each distinct pair i, j, and that x is an umbilic point. Since x is arbitrary, $G_p(r)$ is totally umbilic in M. We therefore get our conclusion by virtue of Theorem 3.3 in [CV]. □

Next, we characterize a space form $M^n(c)$ of curvature c by observing the extrinsic shape of *circles* of positive curvature on geodesic spheres of sufficiently small radius in $M^n(c)$.

Theorem 2. *Let M be a complete Riemannian manifold of dimension greater than 2. Then the following conditions are equivalent*:

(1) *M is a space form.*

(2) *For each point* $p \in M$ *there is a positive number* ϵ_p *such that for every* r *with* $0 < r < \epsilon_p$ *there exists some* $k = k(p, r) \geq 0$ *satisfying the following property*: *Every circle of curvature* k *on a geodesic sphere* $G_p(r)$ *in* M *is a circle of positive curvature in* M.

(3) *For each point* $p \in M$ *there is a positive number* ϵ_p *such that for every* r *with* $0 < r < \epsilon_p$ *there exists some* $k = k(p, r) \geq 0$ *satisfying the following property*: *Every circle of curvature* k *on a geodesic sphere* $G_p(r)$ *in* M *is a plane curve in* M.

(4) *For each point* $p \in M$ *there is a positive number* ϵ_p *such that for every* r *with* $0 < r < \epsilon_p$ *there exists some* $k = k(p, r) \geq 0$ *satisfying the following property*: *Every circle of curvature* k *on a geodesic sphere* $G_p(r)$ *in* M *is a Frenet curve of order* 2 *in the wider sense in* M.

(5) *For each point* $p \in M$ *there is a positive number* ϵ_p *such that for every* r *with* $0 < r < \epsilon_p$ *there exists some* $k = k(p, r) \geq 0$ *satisfying the following property*: *Every circle of curvature* k *on a geodesic sphere* $G_p(r)$ *in* M *is a curve of order* 2 *in* M.

Proof. It is sufficient to show that (5) implies (1). We show $G_p(r)$ is totally umbilic in the ambient space M. When $k = 0$, the proof of Theorem 1 guarantees this, so we suppose $k > 0$. We choose an arbitrary point $x \in G_p(r)$. For a pair of orthonormal tangent vectors $u, v \in T_x G_p(r)$ we take a circle γ of positive curvature k on $G_p(r)$ with initial condition that $\gamma(0) = x, \dot\gamma(0) = u, \nabla_{\dot\gamma}\dot\gamma(0) = kv$, which satisfies the equations $\nabla_{\dot\gamma}\dot\gamma = kY$ and $\nabla_{\dot\gamma}Y = -k\dot\gamma$. By the Gauss formula we have

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + \langle A\dot\gamma, \dot\gamma\rangle N = kY + \langle A\dot\gamma, \dot\gamma\rangle N,$$

which shows that $\|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\| > 0$. Hence, by hypothetis, Lemma 1 tells us that the curve γ is a Frenet curve of proper order 2 in the ambient space M, so that it satisfies the following differential equations:

$$\begin{cases} \widetilde{\nabla}_{\dot\gamma}\dot\gamma = \sqrt{k^2 + \langle A\dot\gamma, \dot\gamma\rangle^2}\, \widetilde{Y} \\ \widetilde{\nabla}_{\dot\gamma}\widetilde{Y} = -\sqrt{k^2 + \langle A\dot\gamma, \dot\gamma\rangle^2}\, \dot\gamma, \end{cases}$$

where $\widetilde{Y} = \big(kY + \langle A\dot\gamma, \dot\gamma\rangle N\big) / \sqrt{k^2 + \langle A\dot\gamma, \dot\gamma\rangle^2}$. By the second equality we obtain

$$\begin{aligned} &\widetilde{\nabla}_{\dot\gamma}\left(\sqrt{k^2 + \langle A\dot\gamma, \dot\gamma\rangle^2}\, \widetilde{Y}\right) \\ &\quad = -\big(k^2 + \langle A\dot\gamma, \dot\gamma\rangle^2\big)\dot\gamma + \frac{\langle A\dot\gamma, \dot\gamma\rangle\langle A\dot\gamma, \dot\gamma\rangle'}{\sqrt{k^2 + \langle A\dot\gamma, \dot\gamma\rangle^2}}\, \widetilde{Y}. \end{aligned}$$

On the other hand, by use of the formulae of Gauss and Weingarten we find

$$\begin{aligned} &\widetilde{\nabla}_{\dot\gamma}\left(\sqrt{k^2 + \langle A\dot\gamma, \dot\gamma\rangle^2}\, \widetilde{Y}\right) \\ &\quad = \widetilde{\nabla}_{\dot\gamma}\big(kY + \langle A\dot\gamma, \dot\gamma\rangle N\big) = -k^2\dot\gamma + k\langle A\dot\gamma, Y\rangle N \\ &\qquad + \langle A\dot\gamma, \dot\gamma\rangle' N - \langle A\dot\gamma, \dot\gamma\rangle A\dot\gamma. \end{aligned}$$

Comparing these two equalities we can see that

$$\begin{aligned} &\langle A\dot\gamma, \dot\gamma\rangle^2\dot\gamma - \langle A\dot\gamma, \dot\gamma\rangle A\dot\gamma - \frac{k\langle A\dot\gamma, \dot\gamma\rangle\langle A\dot\gamma, \dot\gamma\rangle'}{k^2 + \langle A\dot\gamma, \dot\gamma\rangle^2} Y \\ &\quad = \frac{\langle A\dot\gamma, \dot\gamma\rangle^2\langle A\dot\gamma, \dot\gamma\rangle'}{k^2 + \langle A\dot\gamma, \dot\gamma\rangle^2} N - k\langle A\dot\gamma, Y\rangle N - \langle A\dot\gamma, \dot\gamma\rangle' N. \end{aligned}$$

Taking the inner products of both sides of this equality and the vector fields Y and N, we obtain

$$\langle A\dot\gamma, \dot\gamma\rangle\langle A\dot\gamma, Y\rangle + \frac{k\langle A\dot\gamma, \dot\gamma\rangle\langle A\dot\gamma, \dot\gamma\rangle'}{k^2 + \langle A\dot\gamma, \dot\gamma\rangle^2} = 0, \tag{3.1}$$

and

$$\frac{\langle A\dot\gamma, \dot\gamma\rangle^2\langle A\dot\gamma, \dot\gamma\rangle'}{k^2 + \langle A\dot\gamma, \dot\gamma\rangle^2} - k\langle A\dot\gamma, Y\rangle - \langle A\dot\gamma, \dot\gamma\rangle' = 0. \tag{3.2}$$

Since we have

$$\langle A\dot\gamma, \dot\gamma\rangle' = \langle(\nabla_{\dot\gamma}A)\dot\gamma, \dot\gamma\rangle + 2k\langle A\dot\gamma, Y\rangle,$$

because A is symmetric, it follows from (3.1) and (3.2) that

$$\big(3k^2 + \langle A\dot\gamma, \dot\gamma\rangle^2\big)\langle A\dot\gamma, Y\rangle + k\langle(\nabla_{\dot\gamma}A\dot\gamma, \dot\gamma\rangle = 0.$$

Evaluating this equation at $s = 0$, we find

$$\big(3k^2 + \langle Au, u\rangle^2\big)\langle Au, v\rangle + k\langle(\nabla_u A)u, u\rangle = 0. \tag{3.3}$$

As the pair $u, -v$ is also orthonormal, we have

$$-\big(3k^2 + \langle Au, u\rangle^2\big)\langle Au, v\rangle + k\langle(\nabla_u A)u, u\rangle = 0. \tag{3.4}$$

We then see from (3.3) and (3.4) that $\langle Au, v\rangle = 0$ for each orthonormal pair of tangent vectors $u, v \in T_x G_p(r)$ at an arbitrary point $x \in G_p(r)$, so that our geodesic sphere $G_p(r)$ is totally umbilic in the ambient space M. Therefore we get the desirable result. □

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