Arithmetic forms of Selberg zeta functions with applications to prime geodesic theorem

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Abstract: We obtain an arithmetic expression of the Selberg zeta function for cocompact Fuchsian group defined via an indefinite division quaternion algebra over $\mathbf{Q}$. As application to the prime geodesic theorem, we prove certain uniformity of the distribution.

Key words: Quaternion algebra; Selberg zeta function; Prime geodesic theorem.

1. Introduction. Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$ containing -1_2 with finite covolume $v(\Gamma\backslash\mathfrak{H})$, $\mathfrak{H}$ denoting the upper half plane. The Selberg zeta function attached to Γ is defined by

$$Z_\Gamma(s) := \prod_{\{P\}_\Gamma} \prod_{m=0}^{\infty} \left(1 - N(P)^{-s-m}\right), \quad (\mathrm{Re}(s) > 1)$$

where $\{P\}_\Gamma$ runs through all primitive hyperbolic conjugacy classes of Γ with $\mathrm{tr}(P) > 2$, and $N(P) := |\rho|^2$ with ρ the eigenvalue of $P \in \Gamma$ such that $|\rho| > 1$. The chief concern of this paper is to obtain an arithmetic expression of the Selberg zeta function for cocompact Γ defined via an indefinite division quaternion algebra over $\mathbf{Q}$.

Let $B = (a, b/\mathbf{Q})$ be an indefinite division quaternion algebra over $\mathbf{Q}$ with a and b positive integers which are relatively prime and squarefree. We write a typical element of B in the form

$$q = q_0 + q_1\sqrt{a} + q_2\sqrt{b} + q_3\sqrt{a}\sqrt{b},$$

where $q_i \in \mathbf{Q}$ $(i = 0, 1, 2, 3)$. We denote by $q \longmapsto \bar{q}$ the canonical involution of B and put $n(q) = q\bar{q}$, $\mathrm{tr}(q) = q + \bar{q}$. We choose and fix a maximal order $\mathcal{O}$ of B. Let B^1 (resp. $\mathcal{O}^1$) be the group consisting of all elements q of B (resp, $\mathcal{O}$) with $n(q) = 1$. Since the $\mathbf{R}$-algebra $B \otimes_{\mathbf{Q}} \mathbf{R}$ is isomorphic to $M_2(\mathbf{R})$, B^1 is injectively embedded into $\mathrm{SL}_2(\mathbf{R})$ by this isomorphism. The unit group $\mathcal{O}^1$ can be identified with a cocompact discrete subgroup $\Gamma_{\mathcal{O}}$ of $\mathrm{SL}_2(\mathbf{R})$ which is the image of the following injection:

(1.1)
$$\begin{array}{ccc} \mathcal{O}^1 & \hookrightarrow & \mathrm{SL}_2(\mathbf{R}) \\ q & \longmapsto & \begin{pmatrix} q_0 + q_1\sqrt{a} & q_2\sqrt{b} + q_3\sqrt{a}\sqrt{b} \\ q_2\sqrt{b} - q_3\sqrt{a}\sqrt{b} & q_0 - q_1\sqrt{a} \end{pmatrix} \end{array}$$

We write $Z_{\mathcal{O}^1}(s) := Z_{\Gamma_{\mathcal{O}}}(s)$ with this identification. Since B is indefinite over $\mathbf{Q}$, there is a unique maximal order $\mathcal{O}$ of B up to $B^\times$-conjugation. Therefore, $Z_{\mathcal{O}^1}(s)$ depends only on B and not on the choice of $\mathcal{O}$. We simply write $Z_B(s)$ for the Selberg zeta function $Z_{\mathcal{O}^1}(s)$.

For any basis $\{u_i\}$ of $\mathcal{O}$ over $\mathbf{Z}$, set

$$d(B) = |\det(\mathrm{tr}(u_i u_j))|^{1/2}.$$

The number $d(B)$ is independent of the choice of $\mathcal{O}$ and $\{u_i\}$, and equals the product of prime numbers which ramify at $B/\mathbf{Q}$.

Put

$$\mathcal{D} := \{D \in \mathbf{Z}_{>0} \mid D \equiv 0, 1 \pmod 4,\ \text{not a square}\}.$$

Let $\mathfrak{o}$ be an order of $K = \mathbf{Q}(\sqrt{D})$ and $h(\mathfrak{o}) = h(D)$ be the number of classes of proper $\mathfrak{o}$-ideals in the narrow sense. We moreover set

$$\lambda(K) = \prod_{p \mid d(B)} \left(1 - \left(\frac{K}{p}\right)\right),$$

where (K/p) denotes the Artin symbol for $K = \mathbf{Q}(\sqrt{D})$. Let $\varepsilon_D = (\alpha + \beta\sqrt{D})/2$ with (α, β) being the minimal solution of the Pell equation: $x^2 - Dy^2 = 4$. The main theorem of this paper is as follows:

Theorem 1.1. *Let B be a division indefinite quaternion algebra over $\mathbf{Q}$. Then*

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$$Z_B(s)=\prod_{D>0}^{*}\prod_{n=0}^{\infty}\left(1-\varepsilon_D^{-2(s+n)}\right)^{h(D)\lambda(D)},$$

and

$$\frac{Z_B'}{Z_B}(s)=\sum_{D>0}^{*}\sum_{m=1}^{\infty}h(D)\lambda(D)\log\varepsilon_D^2\cdot\frac{\varepsilon_D^{-2ms}}{1-\varepsilon_D^{-2m}},$$

where $\lambda(D)=\lambda(\mathbf{Q}(\sqrt{D}))$ and the symbol $*$ indicates that D runs through all elements in $\mathcal{D}$ satisfying the following conditions.

(Pr-i) $\left(\frac{K}{p}\right)\neq 1$ *for any prime integers $p\mid d(B)$.*

(Pr-ii) $(f(D),d(B))=1$, *where the positive integer $f(D)$ is given by $D=f(D)^2D_K$, D_K being the discriminant of K.*

Remark 1.2. For $\Gamma=\mathrm{SL}(2,\mathbf{Z})$ and its congruence subgroups, Sarnak [S] obtains such an arithmetic form of $Z_\Gamma(s)$.

Remark 1.3. Though for the proof of Theorem 1.1 we have used the theory of optimal embeddings due to Eichler, the theorem would also be deduced from the result of [BJ] and [S1] ([S2]).

Theorem 1.1 has an application of improving the prime geodesic theorem:

$$\pi_\Gamma(x)\sim \mathrm{li}(x)\sim\frac{x}{\log x}, \tag{1.2}$$

where $\pi_\Gamma(x)$ is the number of primitive hyperbolic conjugacy classes P of Γ whose norm $N(P)$ satisfies that $N(P)\le x$, and the relation "$\sim$" means that the quotient of both sides goes to 1 as $x\to\infty$. If we were able to prove

$$\pi_\Gamma(x+y)-\pi_\Gamma(x)\sim \mathrm{li}(x+y)-\mathrm{li}(x)\sim\frac{y}{\log x}, \tag{1.3}$$

then the uniformity of the distribution would be established. An estimate like (1.3) is called the Brun-Titchmarsh type prime geodesic theorem. When $\Gamma=\mathrm{SL}(2,\mathbf{Z})$, Iwaniec [I, Lemma 4] proved that for $x^{(1/2)}(\log x)^2<y<x$

$$\pi_\Gamma(x+y)-\pi_\Gamma(x)\ll y.$$

His method is applicable to our case by our using Theorem 1.1. We prove:

Theorem 1.4. *Let B be a division indefinite quaternion algebra over $\mathbf{Q}$. Put $\pi_B(x)=\pi_{\mathcal{O}^1}(x)$. Then for $x^{(1/2)}(\log x)^2<y<x$, we have*

$$\pi_B(x+y)-\pi_B(x)\ll y. \tag{1.4}$$

The implied constant depends only on B.

Remark 1.5. (a) Theorem 1.4 gives the best possible range of y in view of the multiplicities of the length spectrum in the following sense: It is known that $N(P)$ is a function of $|\,\mathrm{tr}(P)|$ and grows like $|\,\mathrm{tr}(P)|^2$. When $x\in\mathbf{Z}^2=\{n^2\mid n\in\mathbf{Z}\}$, there exist at least $\sqrt{x}$ different P's which satisfy $|\,\mathrm{tr}(P)|^2=x$. It means $\pi_\Gamma(x)$ jumps by as much as $\sqrt{x}$ at that moment. Therefore (1.4) is not true for $y<\sqrt{x}$. Hence the exponent $1/2$ in the lower bound of the range of y in Theorem 1.4 is the best possible.

(b) Theorem 1.4 gives the best possible exponents of x and y according to the conjectural form (1.3).

(c) The current best error term of (1.2) for arithmetic cocompact groups is obtained by Koyama [K]:

$$\pi_B(x)=\mathrm{li}(x)+O(x^{(7/10)+\varepsilon}). \tag{1.5}$$

By using this error term one easily computes that Theorem 1.4 is valid for $x^{(7/10)+\varepsilon}<y<x$. Hence Theorem 1.4 is nontrivial for $x^{(1/2)}(\log x)^2<y\le x^{(7/10)}$.

(d) The estimate (1.5) together with Theorem 1.1 implies the following estimates for class numbers:

$$\sum_{0<\varepsilon_D\le x}^{*}h(D)\lambda(D)=\mathrm{li}(x^2)+O(x^{(7/5)+\varepsilon}),$$

$$\sum_{0<\varepsilon_D\le x}^{*}h(D)\lambda(D)\log\varepsilon_D=\frac{x^2}{2}+O(x^{(7/5)+\varepsilon}),$$

which should be compared with [S, Theorem 4.11] and [H, p. 519, Proposition 2.9].

2. Explicit Form. We introduce the following two theorems due to Eichler.

Theorem 2.1 (Eichler [E])**.** *Let K be a quadratic field over $\mathbf{Q}$ and $\mathfrak{o}_K$ the maximal order of K. Each order $\mathfrak{o}$ of K has an expression: $\mathfrak{o}=\mathbf{Z}+f\mathfrak{o}_K$ for some positive integer $f=f(\mathfrak{o})$. The discriminant of $\mathfrak{o}$ is given by $D(\mathfrak{o}):=f^2D_K$, D_K being the discriminant of K. Then,* (i) *There exists a $\mathbf{Q}$-isomorphism φ of K into B, if and only if $(K/p)\neq 1$ for all prime integers $p\,|\,d(B)$.* (ii) *Let K satisfy the condition of* (i) *and $\mathfrak{o}$ an order of K. Then there exists a $\mathbf{Q}$-isomorphism φ of K into B such that $\varphi(\mathfrak{o})=\varphi(K)\cap\mathcal{O}$, if and only if $(f(\mathfrak{o}),d(B))=1$.*

Denote by $I(K,\mathfrak{o})$ the set of all $\mathbf{Q}$-isomorphisms φ of K into B such that $\varphi(\mathfrak{o})=\varphi(K)\cap\mathcal{O}$. We say that, for φ, $\varphi'\in I(K,\mathfrak{o})$, φ' is $\mathcal{O}^1$-equivalent to φ, if there exists some $\varepsilon \in \mathcal{O}^1$ such that $\varphi'(z) = \varepsilon\varphi(z)\varepsilon^{-1}$ for any $z \in K$. Denote by $I(K, \mathfrak{o})/\mathcal{O}^1$ the set of all the $\mathcal{O}^1$-equivalence classes in $I(K, \mathfrak{o})$.

Theorem 2.2 (Eichler [E]). *We have*

$$\sharp\left(I(K,\mathfrak{o})/\mathcal{O}^1\right) = h(\mathfrak{o})\lambda(K).$$

For a proof we refer to Shimizu [Sh] (see also [A]).

Now we need the relation between the quadratic field over $\mathbf{Q}$ and the quaternion algebra. Set

$$\widetilde{L} := \{x \in \mathbf{Z} + 2\mathcal{O} \mid \mathrm{tr}(x) = 0\}.$$

Any non zero element $x \in \widetilde{L}$ is called primitive, if it cannot be expressed as $x = my$ with $m \in \mathbf{Z}$, $m \neq \pm 1$, $y \in \widetilde{L}$. Denote by $\widetilde{L}_{pr}$ the subset of $\widetilde{L}$ consisting of primitive elements of $\widetilde{L}$. For each positive discriminant D let

$$\mathcal{C}^{pr}(D) := \{\xi \in \widetilde{L}_{pr} \mid n(\xi) = -D\}.$$

In view of Theorem 2.1 we see the following relation:

Lemma 2.3. *It holds that $\mathcal{C}^{pr}(D) \neq \phi$, if and only if D satisfies the conditions* (Pr-i) *and* (Pr-ii).

Proof. For each $x \in C^{pr}(D)$ we form an isomorphism $\varphi_x : K \longrightarrow B$ by $\varphi_x(\sqrt{D}) = x$.

Let $\mathfrak{o}$ be an order of K with discriminant D. We put $x = p + 2\xi$ for $p \in \mathbf{Z}$ and $\xi \in \mathcal{O}$. Because $\mathrm{tr}(x) = 0$, we have $n(x) + p^2 = 4n(\xi)$. From $n(x) = -D$ and $n(\xi) \in \mathbf{Z}$ we have $p^2 \equiv D \pmod 4$.

When $D \equiv 1 \pmod 4$, we have $1 + p \in 2\mathbf{Z}$ and

$$1 + x = 1 + p + 2\xi \in 2\mathbf{Z} + 2\mathcal{O} \subset 2\mathcal{O}.$$

In the case of $D \equiv 0 \pmod 4$, we have $p \in 2\mathbf{Z}$ and

$$x = p + 2\xi \in 2\mathbf{Z} + 2\mathcal{O} \subset 2\mathcal{O}.$$

By the isomorphism φ_x, we have

$$\varphi_x(\mathfrak{o}) = \begin{cases} \mathbf{Z} + \dfrac{1+x}{2}\mathbf{Z} & \text{if } D \equiv 1 \pmod 4, \\[2ex] \mathbf{Z} + \dfrac{x}{2}\mathbf{Z} & \text{if } D \equiv 0 \pmod 4. \end{cases}$$

Then we have $\varphi_x(\mathfrak{o}) \subset \mathcal{O}$. From the primitivity of x, there does not exist $n \geq 2$ which satisfies $\varphi_x(\mathfrak{o}) \subset n\mathcal{O}$. Theorem 2.1 leads to (Pr-i) and (Pr-ii).

Conversely, we assume (Pr-i) and (Pr-ii). From Theorem 2.1, there exists a $\mathbf{Q}$-isomorphism $\varphi : K \to B$. When $K = \mathbf{Q}(\sqrt{D})$, we form $x := \varphi(\sqrt{D})$. Since $\sqrt{D} \in \mathbf{Z} + 2\mathfrak{o}$, we have $x \in \mathbf{Z} + 2\mathcal{O}$. Since $\mathfrak{o}_K$ is the maximal order of K, $\mathfrak{o} = \mathbf{Z} + f(\mathfrak{o})\mathfrak{o}_K$ is given by

(2.1)
$$\mathfrak{o} = \begin{cases} \mathbf{Z} + \dfrac{f(\mathfrak{o}) + \sqrt{D}}{2}\mathbf{Z}, & D_K \equiv 1 \pmod 4, \\[2ex] \mathbf{Z} + \dfrac{\sqrt{D}}{2}\mathbf{Z}, & D_K \equiv 0 \pmod 4. \end{cases}$$

Then since there does not exist $n \geq 2$ such that $(\sqrt{D}/n) \in \mathbf{Z} + 2\mathfrak{o}$, it holds that x is primitive. It follows that $x \in C^{pr}(D)$. □

Set

$$C^{pr} := \bigcup_{D>0}^{*} C^{pr}(D),$$

where D runs over all positive discriminants satisfying the conditions (Pr-i) and (Pr-ii).

Denote by $Prm^+(\mathcal{O}^1)$ the set of primitive elements γ of $\mathcal{O}^1$ with $\mathrm{tr}(\gamma) > 2$. For $\varepsilon \in Prm^+(\mathcal{O}^1)$, we put $\mathbf{Q}(\varepsilon) := \mathbf{Q} + \mathbf{Q}\varepsilon$. Since B is a division quaternion algebra, $\mathbf{Q}(\varepsilon)$ is a quadratic extension over $\mathbf{Q}$ and is isomorphic to $K = \mathbf{Q}(\sqrt{d^2-4})$ over $\mathbf{Q}$ with $d = \mathrm{tr}(\varepsilon)$. We denote this isomorphism by $\varphi : K \longrightarrow \mathbf{Q}(\varepsilon)$ given by $\varphi((d + \sqrt{d^2-4})/2) = \varepsilon$. By our putting $\mathbf{o} := \mathbf{Q}(\varepsilon) \cap \mathcal{O}$ which is an order of $\mathbf{Q}(\varepsilon)$, it holds $\mathfrak{o} := \varphi^{-1}(\mathbf{o})$ is an order of K. One can write $\mathfrak{o} = \mathbf{Z} + f(\mathfrak{o})\mathfrak{o}_K$ with $f(\mathfrak{o}) \in \mathbf{Z}_{>0}$, $\mathfrak{o}_K$ being the maximal order of K. If we set $D = f(\mathfrak{o})^2 D_K$, then D is the discriminant of $\mathfrak{o}$. Since $\varphi(\mathfrak{o}) = \mathbf{o} = \mathbf{Q}(\varepsilon) \cap \mathcal{O}$, Theorem 2.1 implies $(f(\mathfrak{o}), d(B)) = 1$. We see that $\mathcal{C}^{pr}(D) \neq \phi$ for D determined by the order of $\mathbf{Q}(\varepsilon)$.

Lemma 2.4. *It holds that*

$$\varphi^{-1}(\varepsilon) = \varepsilon_D,$$

where D is the discriminant of $\mathfrak{o}$, and $\varepsilon_D = (\alpha + \beta\sqrt{D})/2$ with (α, β) $(\alpha, \beta \in \mathbf{Z}_{>0})$ being the minimal solution of the Pell equation $x^2 - Dy^2 = 4$.

Proof. We have

$$\varphi^{-1}(\varepsilon) = \frac{d + \sqrt{d^2-4}}{2}.$$

We put $\alpha := d$ and $\beta^2 D := d^2 - 4$, where $D = f(\mathfrak{o})^2 D_K$. In what follows we prove (α, β) is the minimal. Assume (α_0, β_0) is the minimal solution, which is not (α, β). Then there exists $n \in \mathbf{Z}(\neq 1)$ such that

$$\frac{\alpha + \beta\sqrt{D}}{2} = \left(\frac{\alpha_0 + \beta_0\sqrt{D}}{2}\right)^n.$$

By the $\mathbf{Q}$-isomorphism φ, we have

$$\varepsilon = \varphi\left(\frac{\alpha+\beta\sqrt{D}}{2}\right) = \varphi\left(\left(\frac{\alpha_0+\beta_0\sqrt{D}}{2}\right)^n\right) = \varphi\left(\frac{\alpha_0+\beta_0\sqrt{D}}{2}\right)^n.$$

This contradicts ε is primitive. □

Now we have $K = \mathbf{Q}(\sqrt{d^2-4}) = \mathbf{Q}(\sqrt{D})$. By using the correspondence in Lemma 2.4, we have the following lemma.

Lemma 2.5. *Let the notation be the same as in Lemma* 2.4. *The map $Prm^+(\mathcal{O}^1) \in \varepsilon \longmapsto \xi \in C^{pr}$, where ξ is given by $\xi = (2\varepsilon-\alpha)/\beta$, is a bijection.*

Proof. Let $\varepsilon \in Prm^+(\mathcal{O}^1)$ be given. We put α, β and D to be the same as in the proof of Lemma 2.4. Set $\xi = (2\varepsilon-\alpha)/\beta$, then we have $\mathrm{tr}(\xi) = 0$. From $n(\varepsilon) = n((\alpha+\beta\xi)/2) = 1$, ξ satisfies $\alpha^2+\beta^2 n(\xi) = 4$. Since (α,β) is the solution of the Pell equation $x^2-Dy^2 = 4$, we have $n(\xi) = -D$. By using

$$\varphi^{-1}(\varepsilon) = \varepsilon_D = \frac{\alpha+\beta\sqrt{D}}{2}$$

as Lemma 2.4, we have

$$\varphi^{-1}(\xi) = \sqrt{D}.$$

The definition of D gives $\sqrt{D} = f(\mathfrak{o})\sqrt{D_K}$. Because of $\sqrt{D_K} \in \mathfrak{o}_K$ and $\mathfrak{o} = \mathbf{Z}+f(\mathfrak{o})\mathfrak{o}_K$, we have $\sqrt{D} \in \mathfrak{o}$.

From (2.2) we get $\sqrt{D} \in \mathbf{Z}+2\mathfrak{o}$. Since ε is a primitive element, (α,β) is the minimal solution. It shows that there does not exist $n \geq 2$ such that $(\sqrt{D}/n) \in \mathbf{Z}+2\mathfrak{o}$. From $\varphi(\mathfrak{o}) = \mathbf{o} = \mathbf{Q}(\varepsilon)\cap\mathcal{O}$ and $\varphi(\sqrt{D}) = \xi$, we have $\xi \in \mathbf{Z}+2(\mathbf{Q}(\varepsilon)\cap\mathcal{O}) \subset \mathbf{Z}+2\mathcal{O}$ and also we deduce that ξ is a primitive element in $\widetilde{L}$. Therefore $\xi \in C^{pr}(D)$.

Conversely, we choose and fix an element ξ in C^{pr} and put $D := -n(\xi)$. Let $(\alpha,\beta) \in \mathbf{Z}\times\mathbf{Z}$ be the minimal solution of the Pell equation $x^2-Dy^2 = 4$, and set $\varepsilon := (\alpha+\beta\xi)/2$. Then we have

$$n(\varepsilon) = \frac{\alpha^2+\beta^2 n(\xi)}{4} = 1,$$

and by $\xi \in \mathbf{Z}+2\mathcal{O}$, we also have

$$\alpha+\beta\xi \in 2\mathcal{O}.$$

Thus we have $\varepsilon \in \mathcal{O}^1$. Since ξ is primitive, there does not exist $n \geq 2$ such that $\varphi^{-1}(\xi/n) = (\sqrt{D}/n) \in \mathbf{Z}+2\mathfrak{o}$. Therefore ε is primitive. This completes the proof. □

We denote by $C^{pr}/\mathcal{O}^1$ (resp. $C^{pr}(D)/\mathcal{O}^1$) the set of $\mathcal{O}^1$-conjugacy classes of C^{pr} (resp. $C^{pr}(D)$).

Lemma 2.6. *The correspondence in Lemma 2.4 induces a bijection of $Prm^+(\mathcal{O}^1)/\mathcal{O}^1$ onto $C^{pr}/\mathcal{O}^1$.*

Proof. Let $\varepsilon, \varepsilon' \in Prm^+(\mathcal{O}^1)$. When ε is $\mathcal{O}^1$-conjugate to ε', there exists $\gamma \in \mathcal{O}^1$ such that $\varepsilon' = \gamma\varepsilon\gamma^{-1}$. Since $\mathbf{Q}(\varepsilon')\cap\mathcal{O} = \gamma(\mathbf{Q}(\varepsilon)\cap\mathcal{O})\gamma^{-1}$ and both of ε, ε' are primitive, the corresponding minimal solutions of the Pell equations are the same. Therefore we may write

$$\varepsilon = \frac{\alpha+\beta\xi}{2} \quad \text{and} \quad \varepsilon' = \frac{\alpha+\beta\xi'}{2}$$

with $\alpha, \beta \in \mathbf{Z}_{>0}$. Thus $\xi' = \gamma\xi\gamma^{-1}$. □

Let $D \in \mathbf{Z}_{>0}$ be a discriminant satisfying the conditions (Pr-i) and (Pr-ii). From Lemma 2.3, we easily see that there exists a bijection from $C^{pr}(D)$ to $I(K,\mathfrak{o})$, where $K = \mathbf{Q}(\sqrt{D})$ and $\mathfrak{o}$ is the order of K with discriminant D. Thus we have:

Lemma 2.7. *There exists a bijection from $C^{pr}(D)/\mathcal{O}^1$ to $I(K,\mathfrak{o})/\mathcal{O}^1$.*

Proof. For $x, x' \in C^{pr}(D)$, take φ_x and $\varphi_{x'} \in I(K,\mathfrak{o})$ such that $\varphi_x(\sqrt{D}) = x$ and $\varphi_{x'}(\sqrt{D}) = x'$. Then for $z = p+q\sqrt{D} \in K$, where $K = \mathbf{Q}(\sqrt{D})$ and $p, q \in \mathbf{Q}$, we have

$$\varphi_x(z) = p+qx \quad \text{and} \quad \varphi_{x'}(z) = p+qx'. \tag{2.2}$$

When x' is $\mathcal{O}^1$-equivalent to x, there exists $\gamma \in \mathcal{O}^1$ such that $x' = \gamma x\gamma^{-1}$. Then we have

$$\gamma\varphi_x(z)\gamma^{-1} = \gamma(p+qx)\gamma^{-1} = p+qx' = \varphi_{x'}(z).$$

Conversely, assume φ_x is $\mathcal{O}^1$-equivalent to $\varphi_{x'}$. The equation $\gamma\varphi_x(z)\gamma^{-1} = p+q\gamma x\gamma^{-1}$ means $\varphi_{x'}(z) = p+qx'$. There exists $\gamma \in \mathcal{O}^1$ such that $\gamma\varphi_x(z)\gamma^{-1} = \varphi_{x'}(z)$. From (2.2), we get $\gamma x\gamma^{-1} = x'$. □

In view of the theorem of Eichler (Theorem 2.2), by applying Lemma 2.7 we have

Proposition 2.8. *Let $D \in \mathbf{Z}_{>0}$ be a discriminant satisfying the conditions* (Pr-i), (Pr-ii). *Then*

$$\sharp\left(C^{pr}(D)/\mathcal{O}^1\right) = h(D)\lambda(D).$$

The eigenvalues λ of the element of $\Gamma_{\mathcal{O}} \subset \mathrm{SL}_2(\mathbf{R})$ associated to $\varepsilon \in \mathcal{O}^1$ by injection (1.1) are given by

$$\lambda = \frac{d \pm \sqrt{d^2-4}}{2},$$

where $d = \mathrm{tr}(\varepsilon)$. Now we write $N_B(\varepsilon)$ for the norm of the element associated to ε. From the correspondence in Lemma 2.5, we have

$$N_B(\varepsilon) = \left(\varphi^{-1}(\varepsilon)\right)^2.$$

Then the Selberg zeta function attached to $\mathcal{O}^1$ is

$$Z_B(s) = \prod_{\varepsilon \in Prm^+(\mathcal{O}^1)/\mathcal{O}^1} \prod_{m=0}^{\infty} (1 - N_B(\varepsilon)^{-s-m}).$$

Lemmas 2.4, 2.6 and Proposition 2.8 show Theorem 1.1.

3. Brun-Titchmarsh type prime geodesic theorem. We introduce the following two theorems.

Theorem 3.1 (Landau [L], p. 196)**.** *Let D be a positive discriminant. Then we have*

$$h(D) = \frac{\sqrt{D}}{\log \varepsilon_D} \sum_{n=1}^{\infty} \frac{\chi_D(n)}{n},$$

where $\chi_D(n) = (D/n)$ is Kronecker's symbol.

Theorem 3.2. *For $0 < Y < t$, put $S(Y,t)$ to be the character sum*

$$S(Y,t) := \sum_{Y \le n \le t} \chi_D(n).$$

Then it holds that

$$\Big| S(Y,t) \Big| \ll |D|^{(1/2)} \log |D|.$$

For a proof we refer to Davenport [D, p. 135].

These estimates lead to the following proposition.

Proposition 3.3. *Let $D \in \mathbf{Z}_{>0}$ be a positive discriminant. Then*

$$h(D) \ll D^{1/2}$$

as $D \to \infty$.

Proof. We estimate $\sum_{n=1}^{\infty} \chi_D(n)/n$ by breaking up the sum into $n < Y$ and $n \ge Y$, Y to be determined.

For the first sum, we use a trivial bound:

$$\left| \sum_{n<Y} \frac{\chi_D(n)}{n} \right| \le \sum_{n<Y} \frac{1}{n} \ll \log Y.$$

On the second sum, since the summation by parts gives

$$\sum_{n \ge Y} \frac{\chi_D(n)}{n} = \int_Y^{\infty} \frac{S(Y,t)}{t^2} dt,$$

Theorem 3.2 leads to

$$\sum_{n \ge Y} \frac{\chi_D(n)}{n} \ll \int_Y^{\infty} \frac{D^{1/2} \log D}{t^2} dt = \frac{D^{1/2} \log D}{Y}.$$

These give

$$\left| \sum_{n=1}^{\infty} \frac{\chi_D(n)}{n} \right| \ll \log Y + \frac{D^{1/2} \log D}{Y}.$$

On taking $Y = D^{1/2}$, we get

$$\left| \sum_{n=1}^{\infty} \frac{\chi_D(n)}{n} \right| \ll \log D.$$

Since $\log \varepsilon_D \gg \log D$ by the definition of ε_D, we have the proposition from Theorem 3.1. □

By using Proposition 3.3 and the following estimates for the divisor function $\tau(u)$ for a positive integer u, Theorem 1.4 will be proved.

Lemma 3.4. *For any $\alpha > 1$ and $x \ge 2$,*

$$\sum_{u < \sqrt{x}} \frac{\tau(u)}{u^{\alpha}} \ll_{\alpha} 1 \quad \textit{and} \quad \sum_{u < \sqrt{x}} \frac{\tau(u)}{u} \ll (\log x)^2.$$

where for the first inequality the implied constant depends only on α.

Lemma 3.5. *We have*

$$\sharp\{n \mid n^2 \equiv 4 \pmod{u^2}, n < u^2\} \ll \tau(u),$$

where u and n are integers.

Proof of Theorem 1.4. Let B, $\mathcal{O}$, and $\mathcal{O}^1$ be the same as before. Set $\Gamma = \mathcal{O}^1 \subset \mathrm{SL}_2(\mathbf{R})$. By the definition of $\pi_B(x)$,

$$\pi_B(x+y) - \pi_B(x) = \sum_{\substack{\varepsilon \\ x < N_B(\varepsilon) \le x+y}} 1,$$

where the sum is taken over $\varepsilon \in Prm^+(\mathcal{O}^1)/\mathcal{O}^1$ with $x < N_B(\varepsilon) \le x+y$. We write this sum in terms of positive discriminants D satisfying the conditions (Pr-i) and (Pr-ii) in Section 1:

$$\pi_B(x+y) - \pi_B(x) = \sideset{}{^*}\sum_{\substack{D>0 \\ \sqrt{x} < \varepsilon_D \le \sqrt{x+y}}} h(D)\lambda(D).$$

Let $t(B)$ denote the number of distinct primes dividing $d(B)$. Then obviously, $\lambda(D) \le 2^{t(B)}$.

We have

$$\begin{aligned} \pi_B(x+y) - \pi_B(x) &\le 2^{t(B)} \sideset{}{^*}\sum_{\substack{D \\ \sqrt{x} < \varepsilon_D \le \sqrt{x+y}}} D^{1/2} \\ &\ll \sum_{\substack{D \\ \sqrt{x} < \varepsilon_D \le \sqrt{x+y}}} D^{1/2}. \end{aligned}$$

The estimate of the right hand side is proved by Iwaniec [I]. We give here a more detailed presentation of that proof. Put $\varepsilon_D = (\alpha + \beta\sqrt{D})/2$ with

$\alpha, \beta \in \mathbf{Z}_{>0}$. From the condition on ε_D, it follows that

$$2\sqrt{x} < \alpha + \beta\sqrt{D} \leq 2\sqrt{x+y}, \tag{3.1}$$

and the inverse of each term gives

$$\frac{2}{\sqrt{x+y}} \leq \alpha - \beta\sqrt{D} < \frac{2}{\sqrt{x}}, \tag{3.2}$$

since (α, β) is a solution of the Pell equation.

From (3.2) we have $\alpha = \beta\sqrt{D} + T$ with

$$\frac{2}{\sqrt{x+y}} \leq T \leq \frac{2}{\sqrt{x}}.$$

By combining this with (3.1), we have

$$\sqrt{x} + \frac{T}{2} < \alpha \leq \sqrt{x+y} + \frac{T}{2}. \tag{3.3}$$

By expanding

$$\sqrt{x+y} = \sqrt{x} + \frac{y}{2\sqrt{x}} + E$$

with E the error term satisfying $E = O(x^{-(3/2)}y^2)$ as $y < x$, (3.3) can be written by

$$\sqrt{x} + \frac{T}{2} < \alpha \leq \sqrt{x} + \frac{T}{2} + \frac{y}{2\sqrt{x}} + E. \tag{3.4}$$

We denote the region of α expressed in (3.4) by $\mathcal{T}$. Then we have

$$\pi_B(x+y) - \pi_B(x) \ll \sum_{\alpha \in \mathcal{T}} \sum_{\substack{\beta \\ \alpha^2 - D\beta^2 = 4}} D^{1/2}.$$

By the Pell equation, we have $D \ll (\alpha/\beta)^2$. Hence

$$\pi_B(x+y) - \pi_B(x) \ll \sqrt{x} \sum_{\beta < 2\sqrt{x}} \frac{1}{\beta} \sum_{\substack{\alpha \in \mathcal{T} \\ \alpha^2 \equiv 4 \pmod{\beta^2}}} 1.$$

The last sum over α is estimated by

$$\tau(\beta)\Big(\frac{1}{\beta^2}\Big(\frac{y}{\sqrt{x}} + E\Big) + 1\Big)$$

from Lemma 3.5. The estimates in Lemma 3.4 now give

$$\pi_B(x+y) - \pi_B(x) \ll y + \frac{y^2}{x} + \sqrt{x}(\log x)^2.$$

It is estimated by y as long as $x^{(1/2)}(\log x)^2 < y < x$. □

Remark 3.6. Zeev Rudnick pointed out that for $1 < y < x$ we can prove

$$\pi_B(x+y) - \pi_B(x) \ll y \log x$$

by omitting the congruence condition at the cost of increasing the number of solutions in the proof of Theorem 1.4.

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References

[A] Arakawa, T.: The dimension of the space of cusp forms on the Siegel upper half plane of degree two related to a quaternion unitary group. J. Math. Soc. Japan, **33**, 126–145 (1981).

[BJ] Bolte, J., and Johansson, S.: A spectral correspondence for Maass waveforms. Geom. Funct. Anal., **9**, 1128–1155 (1999).

[D] Davenport, H.: Multiplicative Number Theory. 2nd ed., Springer, New York (1980).

[E] Eichler, M.: Zur Zahlen Theorie der Quaternionen Algebren. J. reine Angew. Math., **195**, 127–151 (1955).

[H] Hejhal, D.: The Selberg trace formula for PSL(2, **R**). Vol. 2. Lecture Notes in Math., vol. 1001, Springer, New York (1983).

[I] Iwaniec, H.: Prime geodesic theorem. J. reine Angew. Math., **349**, 136–159 (1984).

[K] Koyama, S: Prime geodesic theorem for arithmetic compact manifolds. Internat. Math Res. Notices, no. 8, pp. 383–388 (1998).

[L] Landau, E.: Elementary Number Theory. Chelsea Publishing Co., New York (1958).

[S] Sarnak, P.: Class numbers of indefinite binary quadratic forms. J. Number Theory, **15**, 229–247 (1982).

[Sh] Shimizu, H.: On discontinuous groups operating on the product of the upper half planes. Ann. of Math., **77**, 33–71 (1963).

[S1] Strombergsson, A.: Some remarks on a spectral correspondence of Maass waveforms. Internat. Math Res. Notices, no. 10, pp. 505–517 (2001).

[S2] Strombergsson, A.: An application of an explicit trace formula to a well-known spectral correspondence on quaternion groups. (2000), ([http://www.math.uu.se/˜andreas/papers.html], Preprint).