Examples of globally hypoelliptic operator on special dimensional spheres without infinitesimal transitivity

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Abstract: This paper gives examples of globally hypoelliptic operator on S^3, or on S^7, or on S^{15} which is sum of squares of real vector fields. These operators fail to satisfy the infinitesimal transitivity condition (the Hörmander bracket condition) at every point and therefore they are not hypoelliptic in any subdomain.

Key words: Global hypoellipticity; Omori–Kobayashi conjecture.

1. Introduction. Let M be a closed (compact connected without boundary) $\boldsymbol{C}^\infty$ manifold. For an open subset Ω of M, we denote by $\boldsymbol{C}^\infty(\Omega)$ the space of smooth functions in Ω. A differential operator L is said to be *hypoelliptic* in M if and only if, for any open subset Ω of M, $Lu \in \boldsymbol{C}^\infty(\Omega)$ for a distribution u on M implies $u \in \boldsymbol{C}^\infty(\Omega)$. On the other hand, L is said to be *globally hypoelliptic* on M if and only if $Lu \in \boldsymbol{C}^\infty(M)$ for a distribution u implies $u \in \boldsymbol{C}^\infty(M)$. By definition, hypoelliptic operators are globally hypoelliptic.

Let $Z_1, Z_2, \dots, Z_m$ be smooth real tangent vector fields on M (m is an arbitrary positive integer). The differential operator L which we shall treat is of the form:

$$L = \sum_{j=1}^{m} {Z_j}^* Z_j, \tag{1.1}$$

where ${Z_j}^*$ is the formal adjoint operator of Z_j with respect to a fixed smooth Riemannian metric on M. In this paper, we study a sufficient condition on $Z_1, Z_2, \dots, Z_m$ under which L is globally hypoelliptic on M.

Let $V[Z_1, \dots, Z_m]$ be the linear space defined to be

$$V[Z_1, \dots, Z_m] = \left\{ \sum_{j=1}^{m} f_j Z_j \,;\ f_j \in \boldsymbol{C}^\infty(M) \right\}.$$

For every $Y \in V[Z_1, \dots, Z_m]$, we denote by $\exp tY$ the one parameter diffeomorphism group generated through integral curves by Y, and let $\mathcal{H}[Z_1, \dots, Z_m]$ be the closed subgroup generated by $\{\exp Y \,;\ Y \in V\}$ in the group of $\boldsymbol{C}^\infty$ diffeomorphism of M onto itself.

Definition 1.1. We say that $\mathcal{H}[Z_1, \dots, Z_m]$ is *transitive* on M if there exists, for any $x, y \in M$, a $g \in \mathcal{H}[Z_1, \dots, Z_m]$ such that $x = gy$.

Next, let $\mathcal{L}[Z_1, \dots, Z_m]$ be the Lie algebra generated by $V[Z_1, \dots, Z_m]$.

Definition 1.2. We say that $\mathcal{H}[Z_1, \dots, Z_m]$ is *infinitesimally transitive* at $p \in M$ if $\mathcal{L}[Z_1, \dots, Z_m]|_p = T_pM$.

It is not difficult to see that $\mathcal{H}[Z_1, \dots, Z_m]$ is transitive on M if $\mathcal{H}[Z_1, \dots, Z_m]$ is infinitesimally transitive at every $p \in M$. These geometric notions of transitivity and infinitesimal transitivity are closely related to global hypoellipticity and hypoellipticity. We mention a well-known result due to Hörmander and the conjecture given by Omori and Kobayashi.

Theorem (Hörmander [1]). *If $\mathcal{H}[Z_1, \dots, Z_m]$ is infinitesimally transitive at every $p \in M$, then L defined by* (1.1) *is hypoelliptic in M.*

Conjecture (Omori and Kobayashi [2]). *If $\mathcal{H}[Z_1, \dots, Z_m]$ is transitive on M, then L defined by* (1.1) *is globally hypoelliptic on M.*

Omori and Kobayashi proved this conjecture under an additional condition (the condition **(D)** below).

Now we present an interesting question concerning the above conjecture: "Is it possible to construct a globally hypoelliptic operator L of the form (1.1) with transitive but nowhere infinitesimally transitive system of vector fields $\{Z_1, \dots, Z_m\}$?" As was stud-

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ied in [2], the answer is affirmative in the case where $M = \boldsymbol{T}^3 = [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$. This construction suggests that there will probably exist such a system if M is decomposable to a direct product of three or more closed manifolds. So we are interested in the case where M is not decomposable. In this paper, we demonstrate the existence of such systems on special dimensional spheres S^3, S^7 and S^{15}, where S^m is the m-dimensional standard unit sphere.

Theorem 1.1. *For $n \in \{2, 4, 8\}$, there exist a positive integer $m = m(n)$ and a system of vector fields $\{Z_1, \dots, Z_m\}$ on S^{2n-1} such that the following three conditions hold*:

(A) $\mathcal{H}[Z_1, \dots, Z_m]$ *is transitive on* S^{2n-1}.

(B) *There is no point in S^{2n-1} at which $\mathcal{H}[Z_1, \dots, Z_m]$ is infinitesimally transitive.*

(C) *The differential operator L defined by* (1.1) *is globally hypoelliptic on* S^{2n-1}.

We prove this theorem by constructing $\{Z_1, \dots, Z_m\}$ explicitly. The idea based on [2] is the following. The transitivity of $\mathcal{H}[Z_1, \dots, Z_m]$ implies the a priori estimate

$$\|u\|_0 \le C \|Lu\|_0 + D_N \|u\|_{-N} \quad \text{for all } u \in \boldsymbol{C}^\infty(M),$$

where $\|\cdot\|_s$ stands for the norm of the Sobolev space of order s (see Theorem 2.1 and Corollary 2.4 of [2]). It is not difficult to see that L is globally hypoelliptic on M if we can find a *regulator* Λ, that is, an elliptic pseudodifferential operator of order 1, which commutes with $Z_1, \dots, Z_m$. This fact is generalized by replacing the commutativity condition by the following weaker one introduced in Proposition 3.2 of [2]:

(D) There exists a regulator Λ such that, for every $\delta > 0$ and for all $u \in \boldsymbol{C}^\infty(M)$, the following two estimates hold:

$$\|[\Lambda, L]u\|_{-1} \le \delta \|Lu\|_0 + C_{N,\delta} \|u\|_{-N},$$
$$\|[\Lambda, [\Lambda, L]]u\|_{-2} \le \delta \|Lu\|_0 + C_{N,\delta} \|u\|_{-N}.$$

This condition is trivial if Λ commutes with $Z_1, \dots, Z_m$. The point is that on S^{2n-1} ($n = 2, 4, 8$), we have a globally defined basis $\{W_{jk}^{(n)}\}$ (see §2) which commutes with the Laplacian Δ on S^{2n-1} with respect to the induced metric from $\boldsymbol{R}^{2n}$. For the construction of a system satisfying the conditions in Theorem 1.1, we cut off the support of $W_{jk}^{(n)}$ to reduce the dimension of $\mathcal{L}[Z_1, \dots, Z_m]$, while preserving the transitivity of $\mathcal{H}[Z_1, \dots, Z_m]$ and the condition **(D)** with the regulator $(1 - \Delta)^{1/2}$.

Remark. Let $d(n)$ be the maximal dimension of $\mathcal{L}[Z_1, \dots, Z_m]$ over S^{2n-1}. Then for the systems which we construct, the pair of integers $(m(n), d(n))$ is the following:

$$(m(n), d(n)) = \begin{cases} (3, 2) & (n = 2), \\ (6, 6) & (n = 4), \\ (10, 12) & (n = 8). \end{cases}$$

Notice that $d(n) < 2n - 1$. This means **(B)**.

In §2, we construct a global basis of non-vanishing smooth vector fields on S^{2n-1}. We will take the basis suitably for the study of the transitivity condition **(A)** by using the Hopf mapping. In §3, we present explicit forms of the systems by using these bases. The global hypoellipticity condition **(C)** follows in the same way as [2] by checking the condition **(D)**.

2. The basis of non-vanishing smooth vector fields. Let n be 2 or 4 or 8. Then there exists a global basis of non-vanishing vector fields on S^{2n-1}. We denote by $z = {}^t(\xi, \eta)$ a point of $\boldsymbol{R}^{2n}$, where $\xi, \eta \in \boldsymbol{R}^n$. Here z, ξ and η are column vectors. We construct this basis as restriction of vector fields on $\boldsymbol{R}^{2n}_z$ of the form ${}^t z\, {}^t V \nabla_z$ with an antisymmetric orthogonal matrix V.

We introduce the so-called *Hopf mapping* from $\boldsymbol{R}^{2n}$ to $\boldsymbol{R}^{n+1}$, which turns out to be also from S^{2n-1} to S^n. This enables us to reduce the study of the transitivity on S^{2n-1} to that on S^n and, if we choose the basis of vector fields as follows, to transform the one parameter diffeomorphism groups on S^{2n-1} to rotations on S^n (see (2.7) and (2.8)). We identify $\boldsymbol{R}^n$ with the complex number field $\boldsymbol{C}$ ($n = 2$), the quaternion field $\boldsymbol{H}$ ($n = 4$) or Cayley's algebra $\mathrm{Ca}[\boldsymbol{H}]$ ($n = 8$). The Hopf mapping $\pi^{(n)}$ is defined by

$$\boldsymbol{R}^{2n} \ni z = {}^t(\xi, \eta) \longmapsto \pi^{(n)}(z) = \left(|\xi|^2 - |\eta|^2, 2\xi\eta\right) \in \boldsymbol{R}^{n+1},$$

where $|\xi|$ stands for the Euclidian norm of ξ and $\xi\eta$ the product of ξ and η in the sense of $\boldsymbol{C}$ or $\boldsymbol{H}$ or $\mathrm{Ca}[\boldsymbol{H}]$. We denote the elements by $\pi^{(n)}(z) = (\pi_0^{(n)}(z), \pi_1^{(n)}(z), \dots, \pi_n^{(n)}(z))$. $\pi^{(n)}$ can be regarded as the mapping from S^{2n-1} to S^n, because $|\pi^{(n)}(z)| = |z|^2$.

Each element $\pi_j^{(n)}(z)$ of the Hopf mapping is represented by a real symmetric $2n \times 2n$ matrix $H_j^{(n)}$ as the quadratic form ${}^t z H_j^{(n)} z$ because it is a homogeneous polynomial of degree 2 with respect to z. These matrices are orthogonal and satisfy the following:

(2.1) $$H_j^{(n)} H_k^{(n)} = -H_k^{(n)} H_j^{(n)} \quad (j,k = 0,\ldots,n\,;\ j \neq k).$$

We define new matrices $V_{jk}^{(n)}$ to be

$$V_{jk}^{(n)} = H_j^{(n)} H_k^{(n)} \quad (j,k = 0,\ldots,n\,;\ j \neq k).$$

Then by means of (2.1), we have the following properties of $\{V_{jk}^{(n)}\}$:

(2.2) $$V_{jk}^{(n)} = -V_{kj}^{(n)} \quad \text{if } j \neq k.$$

(2.3) $$V_{jk}^{(n)} V_{kp}^{(n)} = V_{jp}^{(n)}$$ if j, k and p are mutually distinct.

(2.4) $$V_{jk}^{(n)} V_{pq}^{(n)} = V_{pq}^{(n)} V_{jk}^{(n)}$$ if j, k, p and q are mutually distinct.

The basis $W_{jk}^{(n)}$ on S^{2n-1} is defined as the restriction of the vector fields $W_{jk}^{(n)} = {}^t z\, {}^t V_{jk}^{(n)} \nabla$ on $\boldsymbol{R}^{2n}$, where $\nabla = {}^t(\partial_{z_1}, \ldots, \partial_{z_{2n}})$. These vector fields are well-defined on S^{2n-1} thanks to the antisymmetricity (2.2).

By (2.3) and (2.4), we see that $W_{jk}^{(n)}$ have the following relations which we need to observe the dimension of $\mathcal{L}[Z_1, \ldots, Z_m]$:

(2.5) $$[W_{jk}^{(n)}, W_{kp}^{(n)}] = -2W_{jp}$$ if j, k and p are mutually distinct.

(2.6) $$[W_{jk}^{(n)}, W_{pq}^{(n)}] = 0$$ if j, k, p and q are mutually distinct.

On the other hand, the one parameter diffeomorphism group generated by $W_{jk}^{(n)}$ on S^{2n-1} is transformed by $\pi^{(n)}$ to a rotation on S^n:

(2.7) $$\pi_k^{(n)}(\exp(t\, W_{jk}^{(n)})z) = (\cos 2t)\, \pi_k^{(n)}(z) - (\sin 2t)\, \pi_j^{(n)}(z) \quad \text{if } j \neq k.$$

(2.8) $$\pi_p^{(n)}(\exp(t\, W_{jk}^{(n)})z) = \pi_p^{(n)}(z)$$ if j, k and p are mutually distinct.

3. Transitive systems without infinitesimal transitivity. We represent here explicit forms of vector fields satisfying the conditions in Theorem 1.1. We prepare some cut-off functions on S^{2n-1}. Let $\varphi_1(t), \varphi_2(t)$ and $\psi(t)$ be functions on $\boldsymbol{R}$ such that

$$\begin{cases} \varphi_1, \varphi_2, \psi \in \boldsymbol{C}^\infty(\boldsymbol{R}), & 0 \leq \varphi_1, \varphi_2, \psi \leq 1, \\ \varphi_1 = 1 \text{ on } \{t \geq 3/4\}, & \operatorname{supp} \varphi_1 \subset \{t \geq 1/2\}, \\ \varphi_2 = 1 \text{ on } \{t \leq 0\}, & \operatorname{supp} \varphi_2 \subset \{t \leq 1/4\}, \\ \psi = 1 \text{ on } \{t \geq 5/6\}, & \operatorname{supp} \psi \subset \{t > 2/3\}, \end{cases}$$

let $\Phi_1^{(n)}$, $\Phi_2^{(n)}$ $(n = 2,4,8)$ and $\Psi_1^{(n)}$, $\Psi_2^{(n)}$ $(n = 4,8)$ cut-off functions on S^{2n-1} defined as follows:

$$\Phi_1^{(n)}(z) = \varphi_1\left(\pi_0^{(n)}(z)\right),$$

$$\Phi_2^{(n)}(z) = \varphi_2\left(\pi_0^{(n)}(z)\right) \quad (n = 2,4,8),$$

$$\Psi_1^{(4)}(z) = \psi\left(\sum_{j=0}^{1} (\pi_j^{(4)}(z))^2\right),$$

$$\Psi_2^{(4)}(z) = \psi\left(\sum_{j=2}^{4} (\pi_j^{(4)}(z))^2\right),$$

$$\Psi_1^{(8)}(z) = \psi\left(\sum_{j=0}^{3} (\pi_j^{(8)}(z))^2\right),$$

$$\Psi_2^{(8)}(z) = \psi\left(\sum_{j=4}^{8} (\pi_j^{(8)}(z))^2\right).$$

$\Phi_1^{(n)}$ and $\Phi_2^{(n)}$ have their supports near the north pole and on the southern hemisphere respectively. $\Phi_2^{(n)} \Psi_1^{(n)}$ and $\Phi_2^{(n)} \Psi_2^{(n)}$ have their supports on the disjoint domains in the southern hemisphere.

We begin with the case $n = 4, 8$.

Proposition 3.1. *Let n be 4. The following system of six vector fields on S^7 satisfies the conditions* **(A)**, **(B)** *and* **(C)** *in Theorem* 1.1 *:*

$$\Big\{\boldsymbol{W_{04}^{(4)}},\ \boldsymbol{W_{12}^{(4)}},\ \boldsymbol{\Phi_1^{(4)} W_{13}^{(4)}},\ \boldsymbol{\Phi_1^{(4)} W_{23}^{(4)}},\ \boldsymbol{\Phi_2^{(4)} \Psi_1^{(4)} W_{01}^{(4)}},\ \boldsymbol{\Phi_2^{(4)} \Psi_2^{(4)} W_{34}^{(4)}}\Big\}.$$

Proposition 3.2. *Let n be 8. The following system of ten vector fields on S^{15} satisfies the conditions* **(A)**, **(B)** *and* **(C)** *in Theorem* 1.1 *:*

$$\Big\{\boldsymbol{W_{08}^{(8)}},\ \boldsymbol{W_{14}^{(8)}},\ \boldsymbol{W_{25}^{(8)}},\ \boldsymbol{\Phi_1^{(8)} W_{23}^{(8)}},\ \boldsymbol{\Phi_1^{(8)} W_{34}^{(8)}},\ \boldsymbol{\Phi_2^{(8)} W_{37}^{(8)}},\ \boldsymbol{\Phi_2^{(8)} \Psi_1^{(8)} W_{01}^{(8)}},\ \boldsymbol{\Phi_2^{(8)} \Psi_1^{(8)} W_{23}^{(8)}},\ \boldsymbol{\Phi_2^{(8)} \Psi_2^{(8)} W_{67}^{(8)}},\ \boldsymbol{\Phi_2^{(8)} \Psi_2^{(8)} W_{78}^{(8)}}\Big\}.$$

In case $n = 2$, we need another vector field $W^{(2)}$ on $\boldsymbol{R}^4$ which can be regarded as a smooth vector field on S^3:

$$W^{(2)} = {}^t z \begin{pmatrix} O_2 & -I_2 \\ I_2 & O_2 \end{pmatrix} \nabla,$$

where I_2 and O_2 are the 2×2 identity matrix and the 2×2 zero matrix respectively.

Proposition 3.3. *Let n be 2. The following system of three vector fields on S^3 satisfies the conditions* **(A)**, **(B)** *and* **(C)** *in Theorem* 1.1 :

$$\left\{ \boldsymbol{W^{(2)}},\ \boldsymbol{\Phi_1^{(2)} W_{12}^{(2)}},\ \boldsymbol{\Phi_2^{(2)} W_{01}^{(2)}} \right\}.$$

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