Explicit representation of structurally finite entire functions

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Abstract: We say that an entire function is structurally finite if it is constructed from a finite number of quadratic polynomials and exponential functions by Maskit surgeries. In this note, we show that every structurally finite entire function has an explicit representation.

Key words: Structually finite entire functions; Maskit surgery; synthetic deformation spaces.

1. Introduction and the main result. The concept of finite constructability for an entire function, called *structural finiteness*, was introduced in [8], where we stated that structurally finite entire functions have many nice properties, and in particular, permit explicit representation. In this note we give a proof of this representation theorem.

First, we give the definition of structurally finite entire functions.

Definition. We say that an entire function is *structurally finite* if it can be constructed from a finite number of building blocks by Maskit surgeries.

Here a *building block* is either a *quadratic block*:

$$az^2 + bz + c : \mathbf{C} \to \mathbf{C} \quad (a \neq 0)$$

or an *exponential block*:

$$a\exp(bz) + c : \mathbf{C} \to \mathbf{C} \quad (ab \neq 0).$$

We say that a structurally finite entire function is *of type* (p, q) if it is constructed from p quadratic blocks and q exp-blocks.

Next, we say that a point α in $\mathbf{C}$ is a *singular value* of an entire function f if, for every neighborhood U of α, there exists a component V of $f^{-1}(U)$ such that $f : V \to U$ is not biholomorphic. Then a Maskit surgery (by connecting functions) is defined as follows.

Definition. Let $f_j : \mathbf{C} \to \mathbf{C}$ $(j = 1, 2)$ be entire functions, and A_j be the set of all singular values of f_j. Assume that there is a cross cut L in $\mathbf{C}$, i.e. the image L of a proper continuous injection of the real line into $\mathbf{C}$, such that

1. $L \cap A_1$ is coincident with $L \cap A_2$, and is either empty or consists of a single point z_0, which is an isolated point of each A_j,
2. $\mathbf{C} - L$ consists of two connected components D_1 and D_2, where D_j contains $A_j - \{z_0\}$ for each j, and
3. if $L \cap A_1 = L \cap A_2 = \{z_0\}$, then z_0 is a critical value of each f_j: for a small disk U with center z_0 such that $U \cap A_j = \{z_0\}$, $f_j^{-1}(U)$ has a relatively compact component W_j which contains a critical point of f_j for each j.

Under the above assumptions, suppose that an entire function $f : \mathbf{C} \to \mathbf{C}$ satisfies the following condition; there exist

1. components $\tilde{D}_1$ and $\tilde{D}_2$ of $f_1^{-1}(D_2)$ of $f_2^{-1}(D_1)$, respectively, such that $f_j : \tilde{D}_j \to D_{3-j}$ is biholomorphic and $\tilde{D}_j \cap W_j \neq \emptyset$ if $L \cap A_j$ are non-empty,
2. a cross cut $\tilde{L}$ in $\mathbf{C}$ such that f gives a homeomorphism of $\tilde{L}$ onto L, and
3. a conformal map ϕ_j of $\mathbf{C} - \tilde{D}_j$ onto U_j such that $f_j = f \circ \phi_j$ on $\mathbf{C} - \tilde{D}_j$, where U_1 and U_2 are components of $\mathbf{C} - \tilde{L}$.

Then we say that f is constructed from f_1 and f_2 by a *Maskit surgery* with respect to L, and also to $\{W_j\}$ when $L \cap A_j$ are non-empty.

Compare with the Maskit combination for Kleinian groups (cf. [4]). Now we give a proof of the following

Theorem 1 (Representation Theorem). *Every structurally finite entire function has the form*

$$\int^z P(t)e^{Q(t)}dt$$

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with suitable polynomials P and Q.

More precisely, the set of all structurally finite entire functions of type (p,q) is

$$SF_{p,q} = \left\{ \int_0^z (c_p t^p + \cdots + c_0) e^{a_q t^q + \cdots + a_1 t} dt + b \right\},$$

where $c_p a_q \neq 0$ if $q > 0$, and we regard that $SF_{p,0} = \mathrm{Poly}_{p+1}$; the set of all polynomials of degree exactly $p+1$.

Remark. Such primitive functions have already appeared as typical examples in various contexts. See for instance, [1], [2], [3], [5], and [6]. Also recall that Baker [1] first showed that no structurally finite entire functions have wandering domains.

2. Proof of Representaiton Theorem.

We say that a structurally finite entire function of type (p,q) is *simple* if it has $(p+q)$ distinct singular values. First we show that simple functions indeed exist in $SF_{p,q}$.

Example 1.

$$F(z) = \int_0^z P(t) e^{t^q} dt$$

with

$$P(t) = (1 - \epsilon^{n_1} t) \cdots + (1 - \epsilon^{n_p} t)$$

has p critical points

$$\epsilon^{-n_1}, \cdots, \epsilon^{-n_p},$$

and q asymptotic values

$$\left\{ \int_0^\infty P(e^{(2\ell+1)\pi i/q} t) e^{-t^q} e^{(2\ell+1)\pi i/q} dt \right\}_{\ell=0}^{q-1}.$$

Here, if a positive constant ϵ is sufficiently small, then these asymptotic values are mutually distinct. And if $\{n_j\}$ increase rapidly enough, then the critical values, which are real, are also mutually distinct.

Thus such an $F \in SF_{p,q}$ is simple.

Next we show that the family $SF_{p,q}$ is topologically strongly complete.

Definition. We say that a family $\mathcal{F}$ of entire functions is *topologically strongly complete* if every entire function topologically equivalent to an element of $\mathcal{F}$ is actually an element.

Here we say that an entire function g is *topologically equivalent* to another f if there are homeomorphisms φ, ψ of $\mathbf{C}$ onto itself such that $\varphi \circ f = g \circ \psi$.

Then we can show that

Proposition 2. *The family $SF_{p,q}$ of type (p,q) is topologically strongly complete.*

Proof. Suppose that g is topologically equivalent to

$$f(z) = \int^z P(t) e^{Q(t)} dt \in SF_{p,q}.$$

Then g is quasiconformally equivalent to f.

Indeed, let φ, ψ be homeomorphisms of $\mathbf{C}$ onto itself such that $\varphi \circ f = g \circ \psi$. Since the set $\mathrm{sing}(f^{-1})$ of all singular values of f is a finite set, there is an isotopy Φ relative to $\mathrm{sing}(f^{-1})$ which connects φ to a quasiconformal map ψ_2. Then, we can lift Φ to an isotopy $\tilde{\Phi}$ such that $g \circ \tilde{\Phi} = \Phi \circ f$ and that $\tilde{\Phi}$ connects ψ to a quasiconformal map ψ_1 with $\psi_2 \circ f = g \circ \psi_1$. Here we may further assume that ψ_j are normalized, i.e. fix 0 and 1, for every entire function conformally equivalent to f belongs to $SF_{p,q}$.

Now, since f' and g' have the same number of zeros, counted with their multiplicities, we find a polynomial $R(z)$ of degree p such that $g'(z)/R(z)$ has no zeros. Hence we can write $g'(z)$ as $R(z) \exp h(z)$ with an entire function $h(z)$.

Since quasiconformal maps are Hölder continuous, there are some positive numbers $K > 1$ and $A > 1$ such that

$$A^{-1}|z|^{1/K} \leq |\psi_j(z)| \leq A|z|^K$$

for each j. Hence on $\{|z| = r\}$, we have

$$|g(z)| = |\psi_2 \circ f \circ \psi_1^{-1}(z)| \leq A|M(f, A^K r^K)|^K,$$

where $M(f,r) = \max_{|z|=r} |f(z)|$. Since $f \in SF_{p,q}$, and since g and g' have the same order, the function $\log |M(g', r)|$ has a polynomial growth with respect to r. Hence there are some C and N such that

$$|\operatorname{Re} h(z)| \leq C r^N$$

for every z with sufficiently large $r = |z|$, which implies that $h(z)$ is a polynomial.

Finally, let q' be the degree of $h(z)$. Then $g(z)$ has exactly q' finite non-equivalent asymptotic values. Thus we have $q' = q$. □

Now, we will introduce a natural topology on the family of structurally finite entire functions.

Definition. Let f be a non-linear entire function. Then the *full deformation set $FD(f)$* of f is the set of all entire functions g such that there is a quasiconformal self-map ϕ of $\mathbf{C}$ satisfiying the *qc-L^∞ condition*:

$$\|f - g \circ \phi\|_\infty = \sup_{\mathbf{C}} |f - g \circ \phi| < \infty.$$

Here we may assume that *such a ϕ as above is always normalized.*

Definition. For every pair of functions f_1 and f_2 in $FD(f)$, we set

$$d(f_1, f_2) = \inf\left(\log K(\phi_1 \circ \phi_2^{-1}) + \|f_1 \circ \phi_1 - f_2 \circ \phi_2\|_\infty\right),$$

where the infimum is taken over all normalized quasiconformal automorphisms ϕ_1, ϕ_2 of $\mathbf{C}$ satisfying the qc-L^∞ conditions between f and f_1, f_2, respectively.

Proposition 3. *The pseudo-distance d is a distance, and $FD(f)$ with this distance is a complete metric space.*

Definition. We call this distance d on $FD(f)$ the *synthetic Teichmüller distance* on $FD(f)$, and the induced topology the *synthetic Teichmüller topology.*

We can easily see that every element in $SF_{p,q}$ is a structurally finite entire function of type (p, q). And for structurally finite entire functions, we have shown in [9] the following

Theorem 4 (Inclusion Theorem). *For a structurally finite entire function f of type (p, q), the full deformation set $FD(f)$ contains all the structurally finite entire functions of the same type.*

In particular,

$$SF_{p,q} \subset FD(f).$$

Thus the synthetic Teichmüller distance gives a topology on $SF_{p,q}$. Also in the proof of Inclusion Theorem, we have shown the following

Lemma 5. *Two simple structurally finite entire functions of the same type are always mutually topologically equivalent.*

Thus, if the given f is simple, then f is topologically equivalent to F in Example 1. Hence $f \in SF_{p,q}$ by the topological strong completeness of the family $SF_{p,q}$.

Finally, for a general f, we can approximate f by simple functions f_n with respect to the synthetic Teichmüller topology (cf. [9]), by relaxing the relations of singular values.

Lemma 6. *Such a sequence $\{f_n\}$ in $SF_{p,q}$ as above converges to some F_∞ in $SF_{p,q}$ with respect to the synthetic Teichmüller topology. This F_∞ equals f, and hence $f \in SF_{p,q}$.*

Proof. Since $d(f_n, f)$ tend to 0, we may assume that there are normalized quasiconformal maps $\phi_n : \mathbf{C} \to \mathbf{C}$ converging to the identity such that

$$\|f_n \circ \phi_n - f\|_\infty$$

tend to 0. Then f_n are locally uniformly bounded. Hence we may assume that the coefficient vectors of f_n converge, which implies that f_n converge to a function F_∞ in $SF_{p',q'}$ with $p' \leq p, q' \leq q$ locally uniformly.

Then for every $z \in \mathbf{C}$, $f_n(\phi_n(z))$ converge to $F_\infty(z)$. Thus $F_\infty(z) = f(z)$. In particular, F_∞ has q non-equivalent asymptotic values and p critical points counted with their multiplicities. Hence we have that $p' = p, q' = q$, which shows the assertion. □

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