Exceptional surgeries and genera of knots

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Abstract: Let $K(r)$ be the 3-manifold obtained by a Dehn surgery on a hyperbolic knot K in the 3-sphere a along a slope $r \neq \infty$. We show that if $|r| > 3 \cdot 2^{7/4} g$, then $K(r)$ is an irreducible 3-manifold with infinite and word-hyperbolic fundamental group, where g denotes the genus of K.

Key words: Exceptional surgery; hyperbolic 3-manifold.

1. Introduction. The Hyperbolic Dehn Surgery Theorem due to Thurston [11] says that all but finitely many Dehn surgeries on a hyperbolic knot give hyperbolic 3-manifolds. On the number of exceptional cases which can occur, a universal upper bound was obtained in [7, 8].

A Dehn surgery on a knot K is determined by its surgery slope. In particular case that K is a knot in the 3-sphere S^3, such slopes are parameterized by $\mathbf{Q} \cup \{\infty\}$ by using the standard meridian-longitude system. See [10] for detail. With respect to this coordinate, the range of exceptional surgery slopes is unbounded. That is, for any positive number N, there exists a knot in S^3 which admits a Dehn surgery along a slope $r > N$ yielding a non-hyperbolic 3-manifold. See [4, Section 5] for example.

In this paper, we consider the range of exceptional surgery slopes in the coordinate above.

Theorem. *Let $K(r)$ be the closed 3-manifold obtained by a Dehn surgery on a hyperbolic knot K in S^3 along a slope $r \neq \infty$. If $|r| > 3 \cdot 2^{7/4} g$, then $K(r)$ is an irreducible 3-manifold with infinite and word-hyperbolic fundamental group, where g denotes the genus of K.*

It is known that the Thurston's Geometrization Conjecture would imply that irreducible 3-manifolds with infinite and word-hyperbolic fundamental group are actually hyperbolic. See [5, §6] for a survey.

Concerning the surgeries yielding lens spaces, the following conjecture was proposed by Goda and Teragaito in [6].

Conjecture. *If a Dehn surgery on a hyperbolic knot in S^3 along a slope $r \neq \infty$ yields a lens space, then the knot is fibered and $2g + 8 \leq |r| \leq 4g - 1$, where g denotes the genus of the knot.*

They gave an upper bound $12g - 7$ and proved that no such surgeries can occur for genus one knots. As an immediate corollary of our theorem, we obtain a new upper bound $3 \cdot 2^{7/4} g < 10.1g$. This is sharper than theirs when $g \geq 4$. Our argument is quite geometric, and so it is different from theirs completely.

2. Proof. Let M be a 3-manifold with a single toral boundary ∂M. A *slope* r on ∂M means the isotopy class of a non-trivial simple closed curve on ∂M. For a knot in a 3-manifold, the complement of an open tubular neighborhood of the knot is called the *exterior*. When M is the exterior of a knot in S^3, slopes on ∂M are parameterized by $\mathbf{Q} \cup \{\infty\}$ by using the standard meridian-longitude system [10].

Suppose that the interior of M, denoted by $\mathrm{Int}(M)$, admits a complete hyperbolic structure of finite volume. One can take a horoball neighborhood C of the cusp of $\mathrm{Int}(M)$ and then identify ∂M with the boundary ∂C of C. Since ∂C is regarded as a Euclidean torus as demonstrated in [11], the length of a curve on ∂M can be defined. The *length* of a slope r on ∂M is defined as the minimum of the lengths of simple closed curves with slope r, and we denote it by $L(r)$. Note that this length depends upon the choice of C.

Let us prepare the following three lemmas. Let $M(r)$ denote the 3-manifold obtained by Dehn filling along a slope r on ∂M. That is, $M(r)$ denotes the 3-manifold obtained by attaching a solid torus V to M so that a simple closed curve with slope r on ∂M bounds a meridian disk of V. The next lemma was shown by Agol [2], which was also obtained by Lackenby [9].

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Lemma 1 ([2, Lemma 6.1]). *If the length of a slope r on ∂M is greater than 6, then the manifold $M(r)$ is irreducible and its fundamental group is infinite and word-hyperbolic.*

Let us choose a particular horoball neighborhood C as follows. Take a maximal one among those having no overlapping interior, and then slightly shrink it. The next lemma holds for this C, which was given in [1].

Lemma 2 ([1, Theorem 5.3]). *Every slope on ∂M has the length greater than $2^{1/4}$, if M is neither the figure-eight knot exterior, the exterior of the knot 5_2 in the knot table nor the manifold obtained by (2,1)-Dehn-filling on the Whitehead link exterior.*

A properly immersed surface in M is called *essential* if the immersion induces injective maps of the fundamental groups and of the relative fundamental groups. In [2], Agol proved the following.

Lemma 3 ([2, Lemma 5.1]). *Suppose that an essential surface S with boundary in M is given. Let $r_1, \dots, r_n$ be the slopes of boundary components of S. Then $\sum_{i=1}^{n} L(r_i) \leq 6|\chi|$, where χ denotes the Euler characteristic of S.*

Proof of Theorem. We first assume that K is the figure eight knot in S^3. In this case, exceptional surgeries are completely understood, and it is shown in [11] that if $K(r)$ is non-hyperbolic and $r \neq \infty$ then $|r| \leq 4 = 4g$.

Next, in the case that K is the knot 5_2 in S^3, it is also shown in [3] that if $K(r)$ is non-hyperbolic and $r \neq \infty$ then $|r| \leq 4 = 4g$.

Now, we consider a hyperbolic knot K in S^3 neither the figure eight knot nor the knot 5_2. Let M denote the exterior of K. Let p/q be a slope on ∂M, where $p,\ q$ are coprime integers and $q \neq 0$. Suppose that $|p| > 3 \cdot 2^{7/4} g|q|$. By virtue of Lemma 1, we only need to show that $L(p/q) > 6$.

We choose a horoball neighborhood C as above and identify ∂M with ∂C. Let $\widetilde{\partial C}$ be a component of the preimage of ∂C in the universal cover of $\mathrm{Int}(M)$. The preimage of a point on ∂C gives a lattice on $\widetilde{\partial C}$. By fixing the base point O, each primitive lattice point corresponds to a slope on ∂C, and the distance between O and a primitive lattice point is equal to the length of the corresponding slope.

Take a lattice point P such that the path OP is projected to the $|q|$ multiple of the longitude. We can take another primitive lattice point Q corresponding to the slope p/q such that the path PQ is projected to $|p|$ multiple of the meridian. Then, the triangle inequality gives that

$$|p|L(\infty) = PQ \leq OP + OQ = |q|L(0) + L(p/q).$$

This implies that

$$L(p/q) \geq |p|L(\infty) - |q|L(0).$$

Let g be the genus of K, that is, the minimum of the genera of Seifert surfaces for K. Since a minimal genus Seifert surface is essential, $L(0) \leq 6(2g-1)$ holds by Lemma 3.

Combining this and Lemma 2, we conclude

$$L(p/q) > 3 \cdot 2^{7/4} g|q| 2^{1/4} - |q| 6(2g-1) > 6. \quad \square$$

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