A generalization of the weak convergence theorem in Sobolev spaces with application to differential inclusions in a Banach space

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Abstract: The existence theorems for (1) a differential inclusion in a Banach space and (2) a variational problem governed by it are presented. In order to solve this problem, some implications of the weak convergence in the space of vector-valued absolutely continuous functions are also explored.

Key words: Differential inclusion; vector-valued absolutely continuous function; convex normal integrand; lower compactness property.

1. Introduction. Let $\mathfrak{X}$ be a real separable refiexjve Banach space. A correspondence (=multi-valued mapping) $\Gamma : [0,T] \times \mathfrak{X} \twoheadrightarrow \mathfrak{X}$ and a function $u : [0,T] \times \mathfrak{X} \times \mathfrak{X} \to \overline{\mathbf{R}}$ are assumed to be given. A double arrow $\twoheadrightarrow$ indicates the domain and the range of a correspondence. The compact interval $[0,T]$ is endowed with the Lebesgue measure dt. $\mathcal{L}$. denotes the σ-field of the Lebesgue-measurable sets of $[0,T]$.

Let $\mathfrak{W}^{1,p}([0,T],\mathfrak{X})$ be the Sobolev space consisting of functions of $[0,T]$ into $\mathfrak{X}$. And let $\Delta(a)$ be the set of all the solutions in the Sobolev space $\mathfrak{W}^{1,p}([0,T],\mathfrak{X})$ of a differential inclusion:

$$(*) \qquad \dot{x}(t) \in \Gamma(t, x(t)), x(0) = a,$$

where $\dot{x}$ denotes the derivative of x and a is a fixed vector in $\mathfrak{X}$. We consider a variational problem:

$$(\sharp) \qquad \text{Minimize}_{x \in \Delta(a)} \int_0^T u(t, x(t), \dot{x}(t))dt.$$

The object of this paper is to discuss a couple of existence problems as follows:

(i) the existence of a solution for the differential inclusion $(*)$, and

(ii) the existence of an optimal solution for the variational problem $(\sharp)$.

In Maruyama [8][9], I presented a solution of these problems in the special case $\mathfrak{X} = \mathbf{R}^\ell$ by making use of the convenient properties of the weak convergence in the Sobolev space $\mathfrak{W}^{1,2}$ $([0,T], \mathbf{R}^\ell)$; i.e. if a sequence $\{x_n\}$ in $\mathfrak{W}^{1,2}$ $([0,T], \mathbf{R}^\ell)$, weakly converges to some $x^* \in \mathfrak{W}^{1,2}$ $([0,T], \mathbf{R}^\ell)$, then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that

$$(\mathrm{W}) \qquad \begin{aligned} &z_n \to x^* \quad \text{uniformly on } [0,T], \text{ and} \\ &\dot{z}_n \to \dot{x}^* \quad \text{weakly in } \mathfrak{L}^2([0,T], \mathbf{R}^\ell). \end{aligned}$$

However it deserves a special notice that this property does not hold in the space $\mathfrak{W}^{1,2}([0,T],\mathfrak{X})$ if $\dim \mathfrak{X} = \infty$. Taking account of this fact, I provided a new convergence result to overcome this difficulty in the case $\mathfrak{X}$ is a real separable Hilbert space in Maruyama [10]. I also gave a existence theory for the problems (i) and (ii) being based upon this new tool in the framework of a separable Hilbert space.

The purpose of the present paper is a further generalization of my previous results to the case $\mathfrak{X}$ is a real separable refiexive Banach space.

I have also to mention about another improvement added on this occasion. In Maruyama [10], I imposed a very restrictive requirement on the continuity of the correspondence Γ; i.e.

> the correspondence $x \mapsto\!\!\!\!\rightarrow \Gamma(t,x)$ is upper hemi-continuous for each fixed $t \in [0,T]$ with respect to the weak topology for the domain and the strong topology for the range.

I have to admit frankly that this is a very unpleasant assumption. In the present paper, I propose the upper hemi-contimuity of $x \mapsto\!\!\!\!\rightarrow \Gamma(t,x)$ with respect to the "weak-weak" combination of topologies instead of the "weak-strong" combination.

2. A convergence theorem in $\mathfrak{W}^{1,p}([0,T], \mathfrak{X})$. As I have already said, any weakly convergent sequence $\{x_n\}$ in the Sobolev space $\mathfrak{W}^{1,2}([0,T],\mathbf{R}^\ell)$ has a subsequence which satisfies the property (W)

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in section 1.

On the other hand, let $\mathfrak{X}$ be a real Banach space with the Radon-Nikodym property (RNP). Then any absolutely continuous function $f : [0,T] \to \mathfrak{X}$ is Frechet-differentiable a.e. (If the Banach space $\mathfrak{X}$ does not have RNP, this property does not hold. For a counter-example, see Komura [7].) Let $\{x_n\}$ be a sequence in $\mathfrak{W}^{1,p}([0,T],\mathfrak{X})$ which weakly converges to some $x^* \in \mathfrak{W}^{1,p}([0,T],\mathfrak{X})$. We should keep in mind that it is not necessarily true that the sequence $\{x_n\}$ has a subsequence $\{z_n\}$ which satisfies the propery (W) if $\dim \mathfrak{X} = \infty$ even in the case $p = 2$. (See Maruyama [10] for a counter-example.)

The following theorem cultivated to overcome this difficulty is a generalization of Theorem 1 of Maruyama [10]. Henceforth we denote by $\mathfrak{X}_s$ (resp. $\mathfrak{X}_w$) a Banach space $\mathfrak{X}$ endowed with the strong (resp. weak) topology.

Theorem 1. *Let $\mathfrak{X}$ be a real separable reflexive Banach space. And consider a sequence $\{x_n\}$ in the Sobolev space $\mathfrak{W}^{1,p}([0,T],\mathfrak{X})(p \geqq 1)$. Assume that*

(i) *the set $\{x_n(t)\}_{n=1}^{\infty}$ is bounded (and hence relatively compact) in $\mathfrak{X}_w$ for each $t \in [0,T]$, and*

(ii) *there exists some function $\psi \in \mathfrak{L}^p([0,T], 0, +\infty))$ such that*

$$||\dot{x}_n(t)|| \leqq \psi(t) \text{a.e.}$$

Then there exist a subsequence $\{z_n\}$ of $\{x_n\}$ and some function $x^* \in \mathfrak{W}^{1,p}([0,T],\mathfrak{X})$ such that

(a) *$z_n \to x^*$ uniformly in $\mathfrak{X}_w$ on $[0,T]$, and*

(b) *$\dot{z}_n \to \dot{x}^*$ weakly in $\mathfrak{L}^p([0,T],\mathfrak{X})$.*

Remark. Since $\mathfrak{X}$ is separable and reflexive, the following results hold true. Assume that $p \geqq 1$.

[I] $\mathfrak{L}^p([0,T],\mathfrak{X})$ is separable.

[II] $\mathfrak{L}^p([0,T],\mathfrak{X})'$ is isomorphic to $\mathfrak{L}^p([0,T],\mathfrak{X}')$, where $1/p + 1/q = 1$ and "$'$"denotes the dual space.

[III] Any absolutely continuous function $f : [0,T] \to \mathfrak{X}$ is Fréchet-differentiable a.e. and the " fundamental theorem of calculus " , i.e.

$$f(t) = f(0) + \int_0^t \dot{f}(\tau)d\tau \,;\, t \in [0,T]$$

is valid.

Proof of Theorem 1. (a) To start with, we shall show the equicontinuity of $\{x_n\}$. Since ψ is integrable, there exists some $\delta > 0$ for each $\varepsilon > 0$ such that

$$\begin{aligned} ||x_n(t) - x_n(s)|| &\leqq \int_s^t ||\dot{x}_n(\tau)||d\tau \\ &\leqq \int_s^t \psi(\tau)d\tau \leqq \varepsilon \quad \text{for all } n \end{aligned}$$

provided that $|t - s| \leqq \delta$. This proves the equicontinuity of $\{x_n\}$ in the strong topology for $\mathfrak{X}$. Hence $\{x_n\}$ is also equicontinuous in the weak topology.

Taking account of this fact as well as the assumption (i), we can claim, thanks to the Ascoli-Arzelà theorem, that $\{x_n\}$ is relatively compact in $\mathfrak{C}([0,T],\mathfrak{X}_w)$ (the set of continuous functions of $[0,T]$ into $\mathfrak{X}_w$) with respect to the topology of uniform convergence.

By the assumption (i), $\{x_n(0)\}$ is bounded in $\mathfrak{X}$, say $\sup_n ||x_n(0)|| \leqq C < +\infty$. And the assumption (ii) implies that

$$\left|\left|\int_0^t \dot{x}_n(\tau)d\tau\right|\right| \leqq ||\psi||_1 \quad \text{for all} \quad t \in [0,T].$$

Hence

$$\begin{aligned} \sup_n ||x_n(t)|| &= \sup_n \left|\left|x_n(0) + \int_0^t \dot{x}_n(\tau)d\tau\right|\right| \\ &\leqq C + ||\psi||_1 \quad \text{for all} \quad t \in [0,T]. \end{aligned}$$

Thus each x_n can be regarded as a mapping of $[0,T]$ into the set

$$M = \{w \in \mathfrak{X} \mid ||w|| \leqq C + ||\psi||_1\}.$$

The weak topology on M is metrizable because M is bounded and $\mathfrak{X}$ is a separable refiexive Banach space. Hence if we denote by M_w the space M endowed with the weak topology, then the uniform convergence topology on $\mathfrak{C}([0,T],M_w)$ is metrizable.

Since we can regard $\{x_n\}$ as a relatively compact subset of $\mathfrak{C}([0,T],M_w)$, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ which uniformly converges to some $x^* \in \mathfrak{C}([0,T],\mathfrak{X}_w)$.

(b) Since

$$||\dot{y}_n(t)|| \leqq \psi(t) \quad \text{a.e.},$$

the sequence $\{w_n : [0,T] \to \mathfrak{X}\}$ defined by

$$w_n(t) = \frac{\dot{y}_n(t)}{\psi(t)} \,;\, n = 1, 2, \cdots$$

is contained in the unit-ball of $\mathfrak{L}^\infty([0,T],\mathfrak{X})$ which is weak*-compact (as the dual space of $\mathfrak{L}^1([0,T],\mathfrak{X}')$) by Alaogln's theorem. Note that the weak* topology on the unit ball of $\mathfrak{L}^\infty([0,T],\mathfrak{X})$ is metrizable since $\mathfrak{L}^1([0,T],\mathfrak{X}')$ is separable. Hence $\{w_n\}$ has a subsequence $\{w_{n'}\}$ which converges to some $w^* \in$

$\mathfrak{L}^\infty([0,T],\mathfrak{X})$ in the weak* topology. We shall write $\dot{z}_n = \dot{y}_{n'} = \psi \cdot w_{n'}$.

If we define an operator $A : \mathfrak{L}^\infty([0,T],\mathfrak{X}) \to \mathfrak{L}^p([0,T],\mathfrak{X})$ by

$$A : g \mapsto \psi \cdot g,$$

then A is continuous in the weak* topology for $\mathfrak{L}^\infty$ and the weak topology for $\mathfrak{L}^p$. In order to see this, let $\{g_\lambda\}$ be a net in $\mathfrak{L}^\infty([0,T],\mathfrak{X})$ such that $w^*\text{-}\lim_{\lambda g_\lambda} = g^* \in \mathfrak{L}^\infty([0,T],\mathfrak{X})$; i.e.

$$\int_0^T \langle \alpha(t), g_\lambda(t)\rangle dt \to \int_0^T \langle \alpha(t), g^*(t)\rangle dt \quad \text{for all} \quad \alpha \in \mathfrak{L}^1([0,T],\mathfrak{X}').$$

Then it is quite easy to verify that

$$\int_0^T \langle \beta(t), \psi(t) g_\lambda(t)\rangle dt = \int_0^T \langle \psi(t)\beta(t), g_\lambda(t)\rangle dt$$
$$\to \int_0^T \langle \psi(t)\beta(t), g^*(t)\rangle dt$$
$$\text{for all } \beta \in \mathfrak{L}^p([0,T],\mathfrak{X}'),\ \frac{1}{p} + \frac{1}{q} = 1$$

since $\psi \cdot \beta \in \mathfrak{L}^1([0,T],\mathfrak{X}')$. This proves the continuity of A.

Hence

$$(1) \quad \dot{z}_n = \psi \cdot w_{n'} \to \psi \cdot w^* \text{ weakly in } \mathfrak{L}^p([0,T],\mathfrak{X}),$$

which implies

$$(2) \quad \left\langle \theta, \int_s^t \dot{z}_n(\tau) d\tau \right\rangle = \int_s^t \langle \theta, \dot{z}_n(\tau)\rangle d\tau$$
$$\to \int_s^t \langle \theta, \psi(\tau) \cdot w^*(\tau)\rangle d\tau \quad \text{for all} \quad \theta \in \mathfrak{X}'.$$

On the other hand, since

$$z_n(t) - z_n(s) = \int_s^t \dot{z}_n(\tau) d\tau \quad \text{for all} \quad n,$$

and $z_n(t) - z_n(s) \to x^*(t) - x^*(s)$ in $\mathfrak{X}_w$, we get

$$(3) \quad \left\langle \theta, \int_s^t \dot{z}_n(\tau) d\tau \right\rangle = \langle \theta, z_n(t) - z_n(s)\rangle$$
$$\to \langle \theta, x^*(t) - x^*(s)\rangle \quad \text{for all} \quad \theta \in \mathfrak{X}'.$$

(2) and (3) imply the relation

$$\langle \theta, x^*(t) - x^*(s)\rangle = \left\langle \theta, \int_s^t \psi(\tau) \cdot w^*(\tau) d\tau \right\rangle \quad \text{for all} \quad \theta \in \mathfrak{X}',$$

from which we can deduce the equality

$$(4) \quad x^*(t) - x^*(s) = \int_s^t \psi(\tau) \cdot w^*(\tau) d\tau.$$

By (1) and (4), we get the desired result:

$$\dot{z}_n \to \dot{x}^* = \psi \cdot w^* \quad \text{weakly in} \quad \mathfrak{L}^p([0,T],\mathfrak{X}). \quad \square$$

In the proof of our Theorem 1, we are making use of some ideas of Aubin and Cellina [1] (pp. 13–14) as in Maruyama [10].

3. Differential inclusions. Throughout this section, $\mathfrak{X}$ *is assumed to be a real separable reflexive Banach space.*

Let us begin by specifying some assumptions imposed on the correspondence $\Gamma : [0,T] \times \mathfrak{X}_w \twoheadrightarrow \mathfrak{X}_w$. Special attentions should be paid to the fact that both of the domain and the range of Γ are endowed with the weak topology.

Assumption 1. *Γ is compact-convex-valued; i.e. $\Gamma(t,x)$ is a non-empty, compact and convex subset of $\mathfrak{X}_w$ for all $t \in [0,T]$ and all $x \in \mathfrak{X}$.*

Assumption 2. *The correspondence $x \mapsto\!\!\!\to \Gamma(t,x)$ is upper hemi-continuous (abbreviated as u.h.c.) for each fixed $t \in [0,T]$; i.e. for any fixed $(t,x) \in [0,T] \times \mathfrak{X}_w$ and for any neighborhood V of $\Gamma(t,x) \subset \mathfrak{X}_w$, there exists some neighborhood U of x such that $\Gamma(t,z) \subset V$ for all $z \in U$.*

Assumption 3. *The graph of the correspondence $t \mapsto\!\!\!\to \Gamma(t,x)$ is $(\mathcal{L}, \mathcal{B}(\mathfrak{X}_w))$-measurable for each fixed $x \in \mathfrak{X}$ where $\mathcal{B}(\mathfrak{X}_w)$ denotes the Barel σ-field on $\mathfrak{X}_w$. (For the concept of "measurability" of a correspondence, the best reference is Castaing-Valadier [5] Chap. III.)*

Assumption 4. *Γ is $\mathfrak{L}^p$-integrably bounded; i.e. there exists $\psi \in \mathfrak{L}^p([0,T],(0,+\infty))$ $(p>1)$ such that $\Gamma(t,x) \subset S_{\psi(t)}$ for every $(t,x) \in [0,T] \times \mathfrak{X}$, where $S_{\psi(t)}$ is the closed ball in $\mathfrak{X}$ with the center 0 and the radius $\psi(t)$.*

Lemma 1 (Castaing [2]). *Suppose that a correspondence $\Gamma : [0,T] \times \mathfrak{X} \twoheadrightarrow \mathfrak{X}$ satisfies the Assumptions 1-3, and that a function $x : [0,T] \to \mathfrak{X}$ is Bochner-integrable. Then there exists a closed-valued correspondence $\Sigma : [0,T] \mapsto\!\!\!\to \mathfrak{X}_w$ such that*

$$\Sigma(t) \subset \Gamma(t, x(t)) \quad \textit{for all} \quad t \in [0,T],$$

and the graph $G(\Sigma)$ of Σ is $(\mathcal{L}, \mathcal{B}(\mathfrak{X}_w))$-measurable.

We can show the next lemma in a similar way as in Maruyama [10], taking account of [III] of the Remark on page 6.

Lemma 2. *Let A be a non-empty compact and convex set in $\mathfrak{X}_w$, and X a subset of $\mathfrak{W}^{1,p}([0,T],\mathfrak{X})$ $(p>1)$ defined by*

$$X = \{x \in \mathfrak{W}^{1,p} \mid ||\dot{x}(t)|| \leqq \psi(t) \quad a.e., \quad x(0) \in A\},$$

where $\psi \in \mathfrak{L}^p([0,T],(0,+\infty))$*. Then* X *is non-empty convex and compact in* $\mathfrak{X}_w$*.*

We denote by $\mathcal{B}(0;\mathfrak{X}_w)$ a neighborhood base of the zero element of $\mathfrak{X}_w$ which consists of convex sets. The following lemma plays a crucial role in the subsequent arguments although its proof is easy.

Lemma 3. *Suppose that the Assumptions* 1-2 *are satisfied. Let* (t^*,x^*) *be any point of* $[0,T]\times\mathfrak{X}$*. Define, for any* $V \in \mathcal{B}(0;\mathfrak{X}_w)$*, a subset* $K(t^*;x^*,V)$*, of* $[0,T]\times\mathfrak{X}$ *by*

$$\begin{aligned}&K(t^*;x^*,V)\\ &=\{(t,x)\in[0,T]\times\mathfrak{X} | x\in x^*+V, t=t^*\}.\end{aligned}$$

Then we have

$$\Gamma(t^*,x^*)=\cap_{V\in\mathcal{B}(0;\mathfrak{X}_w)}\overline{\mathrm{co}}\,\Gamma(K(t^*;x^*,V)).$$

(*Here we do not have to distinquish the convex closure with respect to the strong topology and that with respect to the weak topology. So I simply denote it by* $\overline{\mathrm{co}}$*.*)

Lemma 4. *Suppose that the Assumptions* 1, 2 *and* 4 (*with* $p>1$) *are satisfied. Let* A *be a nonempty convex compact subset of* $\mathfrak{X}_w$*. Then the set*

$$\begin{aligned}H\equiv\{(a,x,y)\in A\times X\times X \mid \dot{y}(t)\in\Gamma(t,x(t))\\ \text{a.e. and}\quad x(0)=y(0)=a\}\end{aligned}$$

is weakly compact in $A\times X\times X$*.* (*The set* X *is defined in Lemma* 2.)

Sketch of proof. Since we have already known that $A\times X\times X$ is weakly compact in $\mathfrak{X}\times\mathfrak{W}^{1,p}\times\mathfrak{W}^{1,p}$, it is enough to show that H is a weakly closed subset of $A\times X\times X$.

Since $\mathfrak{W}^{1,p}$ is a refiexive Banach space, the dual of which is separable, the weak topology on the bounded set X is metrizable. So we are permitted to use a sequence argument.

Let $\{q_n=(a_n,x_n,y_n)\}$ be a sequence in H which weakly converges to some $q^*=(a^*,x^*,y^*)$ in $A\times X\times X$. We have to show that $q^*\in H$. And it is cnough to check that

$$y^*(t)\in\Gamma(t,x^*(t))\quad\text{a.e.}$$

The set $\{x_n(t)\}$ is relatively compact in $\mathfrak{X}_w$ (for each $t\in[0,T]$) since we have the evaluation:

$$||x_n(t)||\leqq||a||+\int_0^t||\dot{x}_n(\tau)||d\tau\leqq||a||+\int_0^T\psi(\tau)d\tau$$

by the Assumption 4. Hence, thanks to Theorem 1, $\{q_n\}$ has a subsequence (no change in notation) such that

$$\text{(1)}\qquad x_n(t)\to x^*(t)\quad\text{niformly in}\quad\mathfrak{X}_w,\quad\text{and}$$
$$\text{(2)}\qquad \dot{y}_n(t)\to\dot{y}^*(t)\quad\text{weakly in}\quad\mathfrak{L}^p.$$

Then we can show that

$$\text{(3)}\qquad \dot{y}^*(t)\in\overline{\mathrm{co}}\,\Gamma(K(t;x^*(t),V))\quad\text{a.e.}$$

by a similar reasoning as in Maruyama [10] based upon Mazur's Theorem. Since (3) holds true for all $V\in\mathcal{B}(0;\mathfrak{X}_w)$, it follows that

$$\begin{aligned}\text{(4)}\qquad y^*(t)\in\cap_{V\in\mathcal{B}(0;\mathfrak{X}_w)}\overline{\mathrm{co}}\,\Gamma(K(t;x^*(t),V))\\ \Gamma(t,x^*(t))\quad\text{a.e.}\end{aligned}$$

The last equality in (4) comes from Lemma 3. Thus we have proved that $(a^*,x^*,y^*)\in H$. □

We are now going to find out a solution of $(*)$ in the Sobolev space $\mathfrak{W}^{1,p}([0,T],\mathfrak{X})$, $p>1$. Define a set $\Delta(a)$ in $\mathfrak{W}^{1,p}$ by

$$\Delta(a)=\{x\in\mathfrak{W}^{1,p}\mid x\text{ satisfies }(*)\text{ a.e.}\}$$

for a fixed $a\in\mathfrak{X}$.

Theorem 2. *Suppose that the correspondence* Γ *satisfies the Assumptions* 1-4*. Let* A *be a nonempty, convex and compact subset of* $\mathfrak{X}_w$*. Then*

(i) $\Delta(a^*)\neq\emptyset$ *for any* $a^*\in A$*, and*

(ii) *the correspondence* $\Delta: A\to\mathfrak{W}^{1,p}$ *is compact-valued and u.h.c. on* A_w*, in the weak topology for* $\mathfrak{W}^{1,p}$*.*

The proof can be achieved essentially by the same reasoning as in Maruyama [10], based upon preceding lemmas.

Remark. Among other things, the assumption that the set $\Gamma(t,x)$ is always convex is seriously restrictive, especially from the viewpoint of applications. However there seems to be no easy way to wipe out the convexity assumption. (See Tateishi [12].)

Here it may be suggestive for us to glimpse the special case in which Γ is a (single-valued) mapping. A related result was obtained by Szep [11]. (I am indebted to the late Prof. Tosio Kato for this reference.)

Corollary. *Let* $f:[0,T]\times\mathfrak{X}_w\to\mathfrak{X}_w$ *be a* (*single-valued*) *mapping which satisfies the following three conditions.*

(i) *The function* $x\mapsto f(t,x)$ *is continuous for each fixed* $t\in[0,T]$*.*

(ii) *The function* $t\mapsto f(t,x)$ *is measurable for each fixed* $x\in\mathfrak{X}$*.*

(iii) *There exists* $\psi \in \mathfrak{L}^p([0,T],(0,+\infty))$, $p>1$ *such that* $f(t,x) \in S_{\psi(t)}$ *for every* $(t,x) \in [0,T]\times\mathfrak{X}$; *i.e.* $\sup_{x\in\mathfrak{X}} ||f(t,x)|| \leqq \psi(t)$ *for all* $t\in[0,T]$.

Then the differential equation

$$(**) \quad \dot{x} = f(t,x), x(0) = a \text{ (fixed vector in X)}$$

has at least a solution in $\mathfrak{W}^{1,p}([0,T],\mathrm{X})$. *(A solution of* $(**)$ *is a function* $x \in \mathfrak{W}^{1,p}$ *which satisfies* $(**)$ *a.e.)*

4. Variational problem governed by differential inclusion. *Let* $\mathfrak{X}$ *be a real separable reflexive Banach space throughout this section, too.* Assume that $u : [0,T]\times\mathfrak{X}_w\times\mathfrak{X}_s$, $(-\infty,+\infty]$ is a given proper function. Consider a variational problem:

$$(\sharp) \quad \text{Minimize}_{x\in\Delta(a)} J(x) = \int_0^T u(t,x(t),\dot{x}(t))dt,$$

where $\Delta(a)$ is the set of all the solutions of the differential inclusion $(*)$ discussed in the preceding section.

Definition. Let $(\Omega,\mathcal{E},\mu)$ be a measure space, S a topological space, and $\mathfrak{V}$ a real Banach space. A function $f:\Omega\times S\times\mathfrak{V}\to\overline{\mathbf{R}}$ is assumed to be given. We denote by $\mathfrak{M}(\Omega,S)$ the set of all the $(\mathcal{E},\mathcal{B}(S))$-measurable functions of Ω into S. ($\mathcal{B}(S)$ denotes the Borel σ-field on S.) f is said to have the *lower compactness property* if $\{f^-(\omega,\varphi_n(\omega),\theta_n(\omega))\}$ is weakly relatively compact in $\mathfrak{L}^1(\Omega,\overline{\mathbf{R}})$ for any sequence $\{(\varphi_n,\theta_n)\}$ in $\mathfrak{M}(\Omega,S)\times\mathfrak{L}^p(\Omega,\mathfrak{V})$ $(p\geqq 1)$ which satisfies the following three conditions:

(a) $\{\varphi_n\}$ converges in ineasure to some $\varphi^* \in \mathfrak{M}(\Omega,S)$,
(b) $\{\theta_n\}$ converges weakly to some $\theta^*\in\mathfrak{L}^p(\Omega,\mathfrak{V})$, and
(c) there exists some $C<+\infty$ such that

$$\sup_n \int_\Omega f(\omega,\varphi_n(\omega),\theta_n(\omega))d\mu \leqq C.$$

The following theorem is a variation of a result due to Castaing-Clauzure [3] in the spirit of Ioffe [6].

Theorem 3. *Let* $(\Omega,\mathcal{E},\mu)$ *be a finite complete measare space,* S *a metrizable Souslin space, and* $\mathfrak{V}$ *a separable reflexive Banach space. Suppose that a proper function* $f:\Omega\times S\times\mathfrak{V}\to\overline{\mathbf{R}}$ *satisfies the following conditions*:

(i) f *is a normal integrand*; *i.e.*
  (a) f *is* $(\mathcal{E}\otimes\mathcal{B}(S)\otimes\mathcal{B}(\mathfrak{V}),\mathcal{B}(\overline{\mathbf{R}}))$*-measurable, and*
  (b) *the function* $(\xi,v)\mapsto f(\omega,\xi,v)$ *is lower semicontinuous for any fixed* $\omega\in\Omega$,
(ii) *the function* $v\mapsto f(\omega,\xi,v)$ *is convex for any fixed* $(\omega,\xi)\in\Omega\times S$, *and*
(iii) f *has the lower compactness property.*

Let $\{\varphi_n\}$ *be a sequence in* $\mathfrak{M}(\Omega,S)$ *which convarges in measure to some* $\varphi^*\subset\mathfrak{M}(\Omega,S)$, *Let* $\{\theta_n\}$ *be a sequence in* $\mathfrak{L}^p(\Omega,\mathfrak{V})(1\leqq p<+\infty)$ *which converges weakly to some* $\theta^*\in\mathfrak{L}^p(\Omega,\mathfrak{V})$. *Then we have*

$$\int_\Omega f(\omega,\varphi^*(\omega),\theta^*(\omega))d\mu \leqq \liminf_n \int_\Omega f(\omega,\varphi_n(\omega),\theta_n(\omega))d\mu.$$

Remark. 1° A normal integrand $f:\Omega\times S\times\mathfrak{V}\to\overline{\mathbf{R}}$ which also satisfies the condition (ii) is called a *convex normal integrand.*

2° Ioffe [6] established a fundamental theorem on the lower semi-continuity of a nonlinear integral functional as above in the case both of S and $\mathfrak{V}$ are finite dimensional Euclidean spaces. Theorem 3 is an extension of Ioffe's result to the case of a nonlinear integral functional defined on the space of Bochner integrable functions. See also Valadier [13] for some important results based on the theory of Young measures.

Lemma 5. *Suppose that the Assumptions* 1-4 *are satisfied. Let* $\{x_n\}$ *be a sequence in* $\Delta(a)\subset\mathfrak{W}^{1,p}([0,T],\mathfrak{X})$ $(p>1)$. *Let* $u:[0,T]\times\mathfrak{X}_w\times\mathfrak{X}_s\to\overline{\mathbf{R}}$ *be a proper convex normal integrand with the lower compactness property. Then there exist a subsequence* $\{z_n\}$ *of* $\{x_n\}$ *and* $x^*\in\Delta(a)$ *such that*

$$(1) \qquad J(x^*) \leqq \liminf_n J(z_n),$$

where

$$J(x) = \int_0^T u(t,x(t),x(t))dt.$$

Proof. By the Assumption 4, all the images of x_n's are contained in some closed ball $\overline{B}$ with the center 0; i.e.

$$x_n(t)\in\overline{B} \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad n.$$

Hence we may restrict the domain of u to $[0,T]\times\overline{B}_w\times\mathfrak{X}_s$ provided that the sequence $\{x_n\}$ is concerned. Denoting $\overline{u} = u|_{[0,T]\times\overline{B}\times\mathfrak{X}}$, (restriction of u to $[0,T]\times\overline{B}\times\mathfrak{X}$) we have to show that there exist a subsequence $\{z_n\}$ of $\{x_n\}$ and some $x^*\in\Delta(a)$

such that

$$\int_0^T \overline{u}(t, x^*(t), \dot{x}^*(t))dt \leqq \liminf_n \int_0^T \overline{u}(t, z_n(t), \dot{z}_n(t))dt$$

which is equivalent to (1).

The set $\overline{B}$ endowed with the weak topology is metrizable and compact. Hence it is a Polish space. According to Theorem 1, there exist a subsequence $\{z_n\}$ of $\{x_n\}$ and $x^* \in \mathfrak{W}^{1,p}([0,T], \mathfrak{X})$ such that
(a) $z_n \to x^*$ uniformly in $\overline{B}_w$, and
(b) $\dot{z}_n \to \dot{x}^*$ weakly in $\mathfrak{L}^p([0,T], \mathfrak{X})$.

(a) implies, of course, that $z_n \to x^*$ in measure. Thus applying Theorem 3, we obtain the relation

$$\int_0^T \overline{u}(t, x^*(t), \dot{x}^*(t))dt \leqq \liminf_n \int_0^T \overline{u}(t, z_n(t), \dot{z}_n(t))dt.$$

Finally we have to prove that $x^* \in \Delta(a)$. By (a), it follows that

$$\lim_{n\to\infty} \langle z_n(t), \eta(t)\rangle = \langle x^*(t), \eta(t)\rangle$$

for any $t \in [0,T]$ and $\eta \in \mathfrak{L}^p([0,T], \mathfrak{X}')$, where $1/p + 1/q = 1$. Since $z_n(t) \in \overline{B}$, there exists some positive constant $C < \infty$ such that

$$|\langle z_n(t), \eta(t)\rangle| \leqq C||\eta(t)||.$$

Hence we have, by the Dominated Convergence Theorem, that

$$\lim_{n\to\infty} \int_0^T \langle z_n(t), \eta(t)\rangle dt = \int_0^T \langle x^*(t), \eta(t)\rangle dt \quad \text{for any } \eta \in \mathfrak{L}^p([0,T], \mathfrak{X}').$$

This proves that $z_n \to x^*$ weakly in $\mathfrak{L}^p$.

Conbining this result with (b), we can conclude that $\{z_n\}$ weakly converges to x^* in $\mathfrak{W}^{1,p}$. Since $\Delta(a)$ is weakly closed, $x^* \in \Delta(a)$. □

Let $\{x_n\}$ be a minimizing sequence of the problem (♯). Then, by Lemma 5, $\{x_n\}$ has a subsequence (without change of notaion) such that

$$J(x^*) \leqq \liminf_n J(x_n)$$

for some $x^* \in \Delta(a)$. It is also obvious that

$$\inf_{x\in\Delta(a)} J(x) = \liminf_n J(x_n) \leqq J(x^*).$$

Thus we have proved that x^* is a solution of the problem (♯). Summing up —

Theorem 4. *Suppose that the Assumptions 1-4 with $p > 1$ are satisfied for a correspondence $\Gamma : [0,T] \times \mathfrak{X} \twoheadrightarrow \mathfrak{X}$. Furthermore let $u : [0,T] \times \mathfrak{X}_w \times \mathfrak{X}_s \to \overline{\boldsymbol{R}}$ be a convex normal integrand with the lower compactness property. Then the problem (♯) has a solution.*

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