A note on $K3$ surfaces and sphere packings

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Abstract: We study the Mordell-Weil lattices ([9]) of the elliptic $K3$ surfaces which have been introduced by Inose [4] and Kuwata [6]. The point is that the lattices (of rank up to 18) arising this way can be of some interest in terms of sphere packings. In this note, we treat the case of rank 16, 17 or 18, and show that these Mordell-Weil lattices are potentially comparable with the record lattices in these dimensions ([2]). The detailed account is in preparation, which will include the corresponding results for other dimensions as well.

Key words: Mordell-Weil lattice; $K3$ surface; sphere packings.

1. Elliptic $K3$ surfaces of Inose and Kuwata. Recently Kuwata [6] has constructed elliptic $K3$ surfaces with high Mordell-Weil rank (up to rank 18), starting from the Kummer surfaces $Km(A)$ of the product abelian surfaces $A = C_1 \times C_2$, where $C_1$ and $C_2$ are two elliptic curves defined over $k$, an algebraically closed field contained in the field of complex numbers $C$. More precisely, he defines six elliptic $K3$ surfaces $F(n)$ over $P^1$ ($1 \leq n \leq 6$) for a given $A$, with the property that (i) $F(2)$ is isomorphic to $Km(A)$, and (ii) for each divisor $d$ of $n (n = de)$, $F(n)$ is obtained as the base change of $F(e)$ by the map $t \mapsto t^d$ of degree $d$. Then he determines the Mordell-Weil rank $r(n)$ of $F(n)$ by the well-known relation of the Picard number and the Mordell-Weil rank, using Inose’s theorem ([3]) that two $K3$ surfaces $X$, $Y$ have the same Picard number $\rho(X) = \rho(Y)$ if there is a rational map of finite degree between them. The latter implies

$$\rho(F(n)) = \rho(Km(C_1 \times C_2)) = 18 + h,$$

$$h = \text{rk Hom(C_1, C_2)}.$$

Actually Inose [4] has earlier studied the elliptic fibration $F(2)$ on the Kummer surface $Km(C_1 \times C_2)$, and constructed the elliptic $K3$ surface $F(1)$, together with its defining equation:

$$y^2 = x^3 - (3\sqrt{j_1j_2})t^4x + t^5(t^2 + 2t\sqrt{j_1(j_1 - 1)(j_2 - 1)}),$$

where $j_1$, $j_2$ denote the $j$-invariant of $C_1$, $C_2$ respectively. The reducible fibers of $F(1)$ are as follows.

There are two reducible fibers of Kodaira type $II^*$ at $t = 0$ and $\infty$, and no other ones if $j_1 \neq j_2$. If $j_1 = j_2 \neq 0, 1$, there is an additional one of type $I_2$; if $j_1 = j_2 = 1$, then two of type $I_2$; finally if $j_1 = j_2 = 0$, then there is one of type $IV$.

This information determines the structure of trivial lattice of $F(n)$ ($n \leq 6$) to be $U \oplus V^-$, where $U$ is the hyperbolic lattice of rank 2 and $V = V(n)$ is a sum of the root lattices. In case $j_1 \neq j_2$, we have, according as $n = 1, \ldots, 6$,

$$(3)\quad V(n) = U_8 \oplus U_6 \oplus D_4 \oplus A_2^\otimes, \{0\}, \{0\}.$$

In case $j_1 = j_2 \neq 0, 1$ (resp. $= 0$ or $= 0$), $V(n)$ has an additional factor $A_1^\otimes n$ (resp. $A_1^\otimes 2n$ or $A_2^\otimes n$).

In view of (1), the rank $r(n) = \rho(F(n)) - 2 - \text{rk } V(n)$ is given as follows (cf. [6, Th.4.1]): for any $n \leq 6$, we have

$$(4)\quad r(n) = \text{Min} \{4(n - 1), 16\} + h$$

$$\begin{cases}
0 & \text{if } j_1 \neq j_2 \\
3n & \text{if } j_1 = j_2 \neq 0, 1 \\
2n & \text{if } j_1 = j_2 = 0 \text{ or } 1.
\end{cases}$$

Note, in particular, that if $j_1 \neq j_2$ (i.e. if $C_1$ and $C_2$ are not isomorphic), then $F(5)$ and $F(6)$ have no reducible fibres, and we have

$$r(n) = \rho(F(n)) - 2 = 16 + h \quad (n = 5, 6).$$

2. Reconstruction. Recall that the Kummer surface $X = Km(A)$ is a smooth $K3$ surface obtained from the quotient surface $A/\iota_A$ ($\iota_A$: the inversion map of $A$) by resolving the 16 singular points corresponding to the points of order 2 on $A$.

Suppose $A = C_1 \times C_2$ where $C_i$ is defined by the
We consider the linear pencil of cubic curves (in the projective plane with affine coordinates $x_1, x_2$), associated with (6):

$$f_1(x_1)t = f_2(x_2) \quad (t \in \mathbb{P}^1).$$

It has the 9 base points corresponding to the 9 (out of 16) torsion points of order 2 on $A$. By choosing one of them $x_0 = (x_0^1, x_0^2)$, we perform a Weierstrass transformation (cf. [11, 52]). Namely, by transforming the cubic curve (7) into a Weierstrass cubic over $k(t)$ so that $x^0$ is mapped to the point at infinity $O$, we obtain

$$E^{(1)} : y^2 = x^3 + t^2Ax + t^2B(t)$$

where

$$A \in k, \ B(t) = B_0t^2 + B_1t + B_2 \in k[t] \quad (B_0, B_2 \neq 0)$$

are determined in terms of the coefficients of $f_i$. Then we consider the twist of the elliptic curve $E^{(1)}$

$$F^{(1)} : y^2 = x^3 + t^4Ax + t^5B(t)$$

with respect to the quadratic extension $k(\sqrt{t})/k(t)$. This is essentially the same as Inose’s equation (2) and that given by Kuwata [6].

Further let $E^{(n)}(t)$ (resp. $F^{(n)}(t)$) denote the elliptic curve obtained from (8) (resp. (10)) by replacing $t \rightarrow t^n$. In other words, fixing a compatible system of variables $\{t_n(1 \leq n \leq 6)\}$ such that $(t_n)^n = t_1$, $(t_n)^d = t_\infty$ for $n = de$, we define the elliptic curves over $k(t_n)$ by

$$E^{(n)} = E^{(1)} \otimes_{k(t_1)} k(t_n),$$

$$F^{(n)} = F^{(1)} \otimes_{k(t_1)} k(t_n) \quad (1 \leq n \leq 6).$$

We sometimes write $t$ for $t_n$ in dealing with $E^{(n)}/k(t_n)$. In the following, given an elliptic curve $E/k(t)$, we use the same symbol $E$ to denote the elliptic surface over $\mathbb{P}^1$ (the $t$-line). Note that we have

$$E^{(2)} \simeq F^{(2)} \simeq \text{Km}(C_1 \times C_2).$$

Also we have $E^{(4)} \simeq F^{(4)}$ and $E^{(6)} \simeq F^{(6)}$. The elliptic surfaces $F^{(n)}$ are K3 surfaces for all $n \leq 6$, while $E^{(1)}$ and $E^{(3)}$ are rational elliptic surfaces. There is a rational map of degree $d = n/e$

$$\pi = \pi_{n,e} : F^{(n)} \dasharrow F^{(n)}, \quad (x, y, t) \mapsto (x, y, t^d).$$

Now we assume that $C_1$ and $C_2$ are not isogenous (i.e. $j_1 \neq j_2$), and let $h = \text{rk Hom}(C_1, C_2)$ as before. Also we denote by $C_\tau^\text{e}$ the elliptic curve isomorphic to $C/F^\tau$ as a complex torus, $\tau \in \mathbb{C}$ being a point in the upper half plane $\mathcal{H}$.

**Theorem 3.1.** The Mordell-Weil lattice $L^{(5)} = F^{(5)}(k(t))$ is a positive-definite even integral lattice of rank $r = 16 + h$ and minimal norm $\mu = 4$. Its determinant and the center density $\delta$ are given as follows:

(i) If $h = 0$ (i.e. $C_1$ and $C_2$ are not isogenous), we have $r = 16$ and

$$\det L^{(5)} = 5^4/\nu^2, \quad \delta(L^{(5)}) = \nu/25$$

for some $\nu$ dividing $5^2$. Actually we have $\nu = 1$ here (see below).

(ii) If $h = 1$, let $m \geq 2$ be the minimal degree of isogenies $\phi : C_1 \to C_2$ (the assumption $j_1 \neq j_2$ implies that $m \neq 1$). Then $r = 17$ and

$$\det L^{(5)} = 2m \cdot 5^3/\nu^2, \quad \delta(L^{(5)}) = \nu/5 \cdot \sqrt{10m}$$

for some $\nu$ dividing $5^3$.

(iii) If $h = 2$, we may assume that $C_1 = C_\tau$ and $C_2 = C_{m\tau}$ with $\tau \in \mathcal{H}$ satisfying $ar^2 + b\tau + c = 0$ for some integers $a, b, c$ with $\gcd(a, b, c) = 1$, and some positive integer $m \geq 2$ ([7]). Then $r = 18$ and

$$\det L^{(5)} = (4ac - b^2)m^25^2/\nu^2, \quad \delta(L^{(5)}) = \nu/5m \sqrt{4ac - b^2}$$

for some $\nu$ dividing $5^2$.

**Theorem 3.2.** The Mordell-Weil lattice $L^{(6)} = F^{(6)}(k(t))$ is a positive-definite even integral lattice of rank $r = 16 + h$ and minimal norm $\mu = 4$. Its determinant and the center density $\delta$ are given as follows:

(i) If $h = 0$ (i.e. $C_1$ and $C_2$ are not isogenous), we have $r = 16$ and

$$\det L^{(6)} = 6^4/\nu^2, \quad \delta(L^{(6)}) = \nu/36$$

(ii) If $h = 1$, let $m \geq 2$ be the minimal degree of isogenies $\phi : C_1 \to C_2$ (the assumption $j_1 \neq j_2$ implies that $m \neq 1$). Then $r = 17$ and

$$\det L^{(6)} = 2m \cdot 6^3/\nu^2, \quad \delta(L^{(6)}) = \nu/5 \cdot \sqrt{10m}$$

for some $\nu$ dividing $6^3$.
for some $\nu$ dividing $3^2$.

(ii) If $h = 1$, let $m \geq 2$ be the minimal degree of isogenies from $C_1$ to $C_2$ as before. Then $r = 17$ and

$$\det L^{(5)} = 2m \cdot 6^3 / \nu^2, \quad \delta(L^{(5)}) = \nu / 12 \cdot \sqrt[3]{3m}$$

for some $\nu$ dividing $3^2$.

(iii) If $h = 2$, we take $C_1 = C_\tau$ and $C_2 = C_{m\tau}$ with $\tau \in \mathcal{H}$ as before, with some $m \geq 2$. Then $r = 18$ and

$$\det L^{(6)} = (4ac - b^2)m^2 \nu^2 / \nu^2, \quad \delta(L^{(6)}) = \nu / 6m \sqrt{4ac - b^2}$$

for some $\nu$ dividing $3^2$.

**Remark.** According to [2, Table 1.2], the record lattice in dimension $r = 16, 17, 18$ is given by the lattice $\Lambda_\nu$ whose center density is as follows:

$$\delta(\Lambda_{16}) = 1/16, \quad \delta(\Lambda_{17}) = 1/16, \quad \delta(\Lambda_{18}) = 1/8\sqrt{3}.$$ 

Moreover an upper bound $\beta_r$ for center density is known as follows:

$$\beta_{16} = 0.11774, \quad \beta_{17} = 0.14624, \quad \beta_{18} = 0.18629.$$ 

Let us compare the lattices in the above theorems with these data.

First, for Theorem 3.1:

1. In (i), $\nu = 1$ must hold, since otherwise the center density of $L^{(5)}$ will exceed the bound:

$$\delta(L^{(5)})_{|\nu = 1} = 1/10 \sqrt{5} < \delta(\Lambda_{16})$$

$$= 1/16 < \beta_{16} < \delta(L^{(5)})_{|\nu = 5} = 1/5.$$ 

2. In (ii), consider the case $m = 2$. Then $\nu = 1$ must hold again, because

$$\delta(L^{(5)})_{|\nu = 1} = 1/10 \sqrt{3} < \delta(\Lambda_{17})$$

$$= 1/16 < \beta_{17} < \delta(L^{(5)})_{|\nu = 5} = 1/2 \sqrt{3}.$$ 

3. In (iii), consider the case $a = b = c = 1$ and $m = 2$. Then $\nu = 1$ must hold again, because

$$\delta(L^{(5)})_{|\nu = 1} = 1/10 \sqrt{3} < \delta(\Lambda_{18})$$

$$= 1/8 \sqrt{3} < \beta_{18} < \delta(L^{(5)})_{|\nu = 5} = 1/2 \sqrt{3}.$$ 

Next, for Theorem 3.2:

4. In (i), we have

$$\delta(L^{(6)})_{|\nu = 1} = 1/36 < \delta(\Lambda_{16})$$

$$= 1/16 < \delta(L^{(6)})_{|\nu = 3} = 1/12 < \beta_{16}.$$ 

Therefore, if $\nu = 3$, the lattice $L^{(6)}$ would break the record in dimension 16, but this is unlikely.

5. In (ii), consider the case $m = 2$. Then we have

$$\delta(L^{(6)})_{|\nu = 1} = 1/12 \sqrt{5} < \delta(\Lambda_{17})$$

$$= 1/16 < \delta(L^{(6)})_{|\nu = 3} = 1/4 \sqrt{6} < \beta_{17}.$$ 

Hence, if $\nu = 3$, the lattice $L^{(6)}$ would break the record in dimension 17.

6. In (iii), consider the case $a = b = c = 1$ and $m = 2$. Then

$$\delta(L^{(6)})_{|\nu = 1} = 1/12 \sqrt{3} < \delta(\Lambda_{18})$$

$$= 1/8 \sqrt{3} < \delta(L^{(6)})_{|\nu = 3} = 1/4 \sqrt{3} < \beta_{18}.$$ 

Here again, if $\nu = 3$, the lattice $L^{(6)}$ would break the record in dimension 18.

**4. Outline of the proof.** The elliptic K3 surface $X = F^{(n)} (n = 5, 6)$ has no reducible fibres under the assumption $j_1 \neq j_2$.

As for the minimal norm, this implies that the Mordell-Weil lattice coincides with the narrow Mordell-Weil lattice (see [9], [10] for this and other facts on MWL). Then the height formula says that

$$\langle P, P \rangle = 2X + 2(PO) \geq 4$$

for any $P \in F^{(n)}(k(l))$, $p \neq O$, which shows that the minimal norm $\mu$ of $L^{(n)}$ is at least 4 (cf. [9, Th.8.7]). That $\mu = 4$ follows from the following:

**Proposition 4.1.** The lattice $L^{(n)} (n = 5, 6)$ contains a sublattice isomorphic to $E_8[2]$ whose minimal norm is equal to 4.

N. B. Given a lattice $L$, we denote by $L[n]$ the lattice whose pairings are $n$-times the original pairing on $L$. The letters $A_n, D_n, \ldots, E_8$ stand for the root lattices as usual.

The proof of Proposition 4.1 for $n = 6$ reduces to (ii) of the next proposition, by using the natural map of degree 2 from $F^{(6)}$ to $E^{(3)}$ in view of [9, Prop.8.12]. The case $n = 5$ in similar.

**Proposition 4.2.**

(i) The Mordell-Weil lattice of the rational elliptic surface $E^{(1)}$ is isomorphic to $(A_2 \oplus 2)$, where $A_2$ is the dual lattice of the root lattice $A_2$.

(ii) The Mordell-Weil lattice of the rational elliptic surface $E^{(3)}$ is isomorphic to the root lattice $E_8$ (cf. [8]).

On the other hand, for the determinant, we have

(14) $\det L^{(n)} = \det NS(F^{(n)}) = \det T(F^{(n)})$,

where $T(X)$ denotes the lattice of transcendental
cycles on a complex algebraic surface $X$. The first equality holds because there are no reducible fibres, while the second equality results from the following: by definition, $T(X)$ is the orthogonal complement of $NS(X)$ in $H^2(X, \mathbb{Z})$, which is (modulo torsion) a unimodular lattice by the Poincaré duality. Let $\lambda(X) = \text{tr}T(X) = b_2(X) - \rho(X)$.

Based on Inose’s work [3], [4], we can prove the following.

**Lemma 4.3.** If $X, Y$ are complex surfaces with the same geometric genus $p_g(X) = p_g(Y)$ and if $\pi : X \rightarrow Y$ is a rational map of degree $n$, then we have $\lambda(X) = \lambda(Y)$, and

$$nT(X) \subset \pi^*T(Y) \subset T(X), \quad \pi^*T(Y) \simeq T(Y)[n].$$

**Theorem 4.4.** Let $A = C_1 \times C_2$ and let $F^{(n)}$ be the elliptic K3 surfaces defined in §2. Then, for any $n \leq 6$, $T(F^{(n)})$ has rank $\lambda = 4 - h$ and it contains $T(A)[n]$ as a sublattice of finite index $\nu$ which divides a power of $n$. Thus we have

$$\text{det} T(X) = \text{det} T(Y) \cdot n^{\lambda(X)/\nu^2}. \quad (15)$$

Moreover $\nu$ is equal to 1 if $n = 1$ or 2.

Thus we have outlined the proof of Theorems 3.1 and 3.2.

Of course, Mordell-Weil lattices contain more information on the rational points, not just the minimal norm or determinant. Here are some examples.

**Example.** (1) Take $C_1 : y^2 = x^3 - 1$, $C_2 : y^2 = x^3 - x$ so that $j_1 = 0$, $j_2 = 1$ and $h = 0$. Then $\text{Km}(C_1 \times C_2)$ is isomorphic to the elliptic surface $F^{(2)} : y^2 = x^3 + t^4(t^4 + 1)$. Its Mordell-Weil lattice is $F^{(2)}(k(t)) \simeq (A_2^3[2])^{\oplus 2}$. It has 12 rational points $(x, y) = (at^2, \pm t^2(t^2 \pm 1)^2)$ $(a^3 = \pm 2)$ corresponding to the minimal vectors.

The Mordell-Weil lattice of $F^{(6)} : y^2 = x^3 + t^{12} + 1$ has rank 16. We have

$$F^{(6)}(k(t)) \supset E^{(3)}(k(t^2)) \oplus F^{(3)}(k(t^2)) \supset E_k[2] \oplus (D_4^*[6])^{\oplus 2},$$

where the root lattices come from the Mordell-Weil lattices of rational elliptic surfaces. The generators (even basis) for this sublattice can be explicitly described.

In this case, $\text{det} F^{(6)}(k(t)) = 36^2$ has been directly shown in Usui [12, II].

(2) Next take $C_1 : y^2 = x^3 - 1$ and let $\phi : C_1 \rightarrow C_2$ be an isogeny of degree 2. We have $h = 2$ in this case. Then $\text{Km}(C_1 \times C_2)$ is isomorphic to the elliptic surface $F^{(2)} : y^2 = x^3 + t^4(t^4 - 11t^2 - 1)$, as given by Kuwata.

The Mordell-Weil lattice of $F^{(6)} : y^2 = x^3 + t^{12} - 11t^6 - 1$ has rank $16 + h = 18$. It has a sublattice of rank 16 which is similar to the above. The missing rank comes from $F^{(1)} : y^2 = x^3 + t^6(t^6 - 1)$. In [1], the authors have found rational points (sections) of height 4 on the latter.

Let $\Gamma \subset A = C_1 \times C_2$ be the graph of the isogeny $\phi : C_1 \rightarrow C_2$. By keeping track of the image of $\Gamma$ under the rational map $A \rightarrow \text{Km}(A) \rightarrow F^{(1)}$, these rational points can be found in a slightly more conceptual way.

More generally, $F^{(1)}$ is an interesting K3 surface related to $\text{Hom}(C_1, C_2)$.

**Proposition 4.5.** $\text{Hom}(C_1, C_2)$ has the structure of an even lattice, with norm defined by $\phi \mapsto 2\text{deg}(\phi)$. The Mordell-Weil lattice of the elliptic K3 surface $F^{(1)}$ has the same rank and determinant as the lattice $\text{Hom}(C_1, C_2)$, provided that $j_1 \neq j_2$.

**References**


