Asymptotic behaviour of length spectrum of circles on non-flat complex space forms

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Abstract: In this paper, we study length spectrum of circles on a complex projective space and on a complex hyperbolic space. In particular, we focus ourselves on the asymptotic behaviour of the number of congruency classes of circles with length λ and on the asymptotic behaviour of the number of congruency classes of circles of prescribed geodesic curvature with length not greater than λ.

Key words: Length spectrum; circle; complex space form.

Introduction. Let M be a complete Riemannian manifold. A smooth curve $\gamma : \mathbf{R} \to M$ parametrized by its arclength is called a *circle* of geodesic curvature κ if it satisfies

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma(t) = -\kappa^2\dot\gamma(t).$$

Here κ is a non-negative constant and $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection. In preceding papers [A] and [AM], S. Maeda and the author studied lengths of circles on non-flat complex space forms, which are complex projective spaces and complex hyperbolic spaces. The aim of this paper is to add some results to their papers.

A circle γ is said to be closed if there exists positive t_c with $\gamma(t) = \gamma(t+t_c)$ for every t. The minimum positive t_c with this property is called the length of γ and is denoted by length(γ). For an open circle γ, a circle which is not closed, we put length$(\gamma) = \infty$. We are interested in how lengths of circles are distributed on the real line. In order to get rid of the influence of the isometry group of the base manifold M, we consider congruency classes of circles. We call two circles γ_1 and γ_2 on M *congruent* if there exist an isometry φ of M and a constant t_0 satisfying $\gamma_1(t) = \varphi \circ \gamma_2(t + t_0)$ for all t.

For a complex projective space $\mathbf{C}P^n$ and a complex hyperbolic space $\mathbf{C}H^n$ of complex dimension n (≥ 2), we showed in the preceding papers that for each positive λ there is at least one circle with length λ. So we are interested in quantitative properties. In section 1, we consider the multiplicity at λ, that is the number of congruency classes of circles with length λ, and study its asymptotic behaviour when λ goes to infinity by counting pairs of relatively prime positive integers with some conditions.

In section 2, we classify circles by their geodesic curvature. A circle of null geodesic curvature is nothing but a geodesic. It is well known that for a compact Riemannian manifold of negative curvature there is a resemblance between length of closed geodesics and prime numbers (see [PP], for example). Although our situation is much simpler than this case, we prepare a corresponding result for circles of prescribed geodesic curvature on non-flat complex space forms. We consider the number of congruency classes of circles of given geodesic curvature κ whose length is not greater than λ, and study its asymptotic behaviour when λ goes to infinity.

1. Multiplicity of length spectrum of circles. For a complete Riemannian manifold M, we denote by $\mathrm{Cir}(M)$ the set of all congruency classes of circles on M. The *length spectrum* of circles $\mathcal{L}_M : \mathrm{Cir}(M) \to \mathbf{R} \cup \{\infty\}$ is defined by $\mathcal{L}_M([\gamma]) =$ length(γ), where $[\gamma]$ denotes the congruency class containing γ. We also call the set $\mathrm{LSpec}(M) = \mathcal{L}_M(\mathrm{Cir}(M)) \cap \mathbf{R}$ the length spectrum of circles on M. For example, the length spectrums of real space forms, that is a standard sphere $S^n(c)$ of curvature c, a Euclidean space $\mathbf{R}^n$, and a hyperbolic space $H^n(-c)$ of curvature $-c$, are

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$$\mathrm{LSpec}(S^n(c)) = (0, 2\pi/\sqrt{c}\,],$$
$$\mathrm{LSpec}(\mathbf{R}^n) = \mathrm{LSpec}(H^n(-c)) = (0,\infty).$$

For a complex hyperbolic space $\mathbf{C}H^n(-c)$ of holomorphic sectional curvature $-c$ and a complex projective space $\mathbf{C}P^n(c)$ of holomorphic sectional curvature c whose complex dimension n is not less than 2, their length spectrums are set theoretically simple:

$$\mathrm{LSpec}(\mathbf{C}P^n(c)) = \mathrm{LSpec}(\mathbf{C}H^n(-c)) = (0,\infty).$$

We hence study the number of congruency classes of circles with given length spectrum. We call the cardinality $m_M(\lambda)$ of the set $\mathcal{L}_M^{-1}(\lambda)$ the *multiplicity* of $\mathcal{L}_M$ at a length spectrum λ. For example, for real space forms their multiplicities satisfy

$$m_{S^n(c)}(\lambda) = 1 \text{ for } 0<\lambda\le 2\pi/\sqrt{c},$$
$$m_{S^n(c)}(\lambda) = 0 \text{ for } \lambda > 2\pi/\sqrt{c},$$
$$m_{\mathbf{R}^n}(\lambda) = m_{H^n(-c)}(\lambda) = 1 \text{ for each } \lambda>0.$$

It is much complicated for length spectrums of circles on non-flat complex space forms. In [A] and [AM], we showed the following on multiplicities: When $n \geq 2$,

1) $m_{\mathbf{C}P^n(c)}(\lambda)$ and $m_{\mathbf{C}H^n(-c)}(\lambda)$ are finite at each point λ, but they are not uniformly finite with respect to λ,
2) $m_{\mathbf{C}H^n(-c)}$ is left continuous and monotone increasing.

In this section we study asymptotic behaviour of the multiplicities $m_{\mathbf{C}P^n(c)}(\lambda)$ and $m_{\mathbf{C}H^n(-c)}(\lambda)$ when λ goes to infinity.

Theorem 1. *For a complex projective space $\mathbf{C}P^n(c)$ of constant holomorphic sectional curvature c and complex dimension n (≥ 2), the multiplicity $m_{\mathbf{C}P^n(c)}$ of length spectrum of circles satisfies*

$$\lim_{\lambda\to\infty}\frac{m_{\mathbf{C}P^n(c)}(\lambda)}{\lambda^2\log\lambda} = \frac{9c}{8\pi^4}.$$

Proof. Following [AM] the length spectrum of circles on $\mathbf{C}P^n(c)$ is as follows.

$$\mathrm{LSpec}(\mathbf{C}P^n(c)) = \left(0,\frac{2\pi}{\sqrt{c}}\right)\cup\left(0,\frac{4\pi}{\sqrt{c}}\right) \cup\bigcup\left\{ I_{\tau(p,q)}\;\middle|\; \begin{array}{l} p \text{ and } q \text{ are relatively prime}\\ \text{positive integers with } p>q\end{array}\right\},$$

where the interval $I_{\tau(p,q)}$ is given by

$$I_{\tau(p,q)} = \begin{cases} \left(\dfrac{4\pi}{3\sqrt{c}}\sqrt{2q(3p+q)},\ \dfrac{4\pi}{3\sqrt{c}}\sqrt{9p^2-q^2}\right), & \text{if } pq \text{ is even},\\[2ex] \left(\dfrac{2\pi}{3\sqrt{c}}\sqrt{2q(3p+q)},\ \dfrac{2\pi}{3\sqrt{c}}\sqrt{9p^2-q^2}\right), & \text{if } pq \text{ is odd},\end{cases}$$

and the multiplicity $m_{\mathbf{C}P^n(c)}(\lambda)$ hence coincides with the number of sets $I_{\tau(p,q)}$ containing λ when $\lambda \geq 4\pi/\sqrt{c}$. Therefore what we have to do is to study the cardinality of the set

$$S = \left\{ (p,q)\;\middle|\; \begin{array}{l} p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which satisfy}\\ p>q,\ pq \text{ is even},\\ (4\pi/3)\sqrt{2q(3p+q)/c}<\lambda\\ (4\pi/3)\sqrt{(9p^2-q^2)/c}>\lambda \end{array}\right\} \bigcup \left\{ (p,q)\;\middle|\; \begin{array}{l} p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which satisfy}\\ p>q,\ pq \text{ is odd},\\ (2\pi/3)\sqrt{2q(3p+q)/c}<\lambda\\ (2\pi/3)\sqrt{(9p^2-q^2)/c}>\lambda \end{array}\right\}.$$

For a positive real number λ and a positive integer k, we denote by $a(\lambda)$ and by $a(\lambda;k)$ the cardinality of the sets

$$A(\lambda) = \left\{ (p,q)\in\mathbf{Z}\times\mathbf{Z}\;\middle|\; \begin{array}{l} p \text{ and } q \text{ are relatively}\\ \text{prime integers with}\\ 9p^2-q^2>2\lambda^2,\\ q(3p+q)<\lambda^2,\\ p>q\geq 1\end{array}\right\},$$

$$A(\lambda;k) = \left\{ (p,q)\in k\mathbf{Z}\times k\mathbf{Z}\;\middle|\; \begin{array}{l} 9p^2-q^2>2\lambda^2,\\ q(3p+q)<\lambda^2,\\ p>q\geq 1\end{array}\right\}$$

respectively. Here $k\mathbf{Z}$ denotes the set $\{kj \mid j\in\mathbf{Z}\}$. Since the correspondence $(p,q)\mapsto(kp,kq)$ of $A(\lambda/k;1)$ to $A(\lambda;k)$ is bijective, we find the following relation by using the Möbius function μ;

$$a(\lambda) = \sum_{k=1}^{\infty}\mu(k)a(\lambda;k) = \sum_{k=1}^{[\lambda/2]}\mu(k)a\left(\frac{\lambda}{k};1\right),$$

where $[\delta]$ denotes the integer part of a real number δ. Since the area of the set

$$\left\{(x,y)\in\mathbf{R}^2\;\middle|\; \begin{array}{c} 9x^2-y^2\geq 2\lambda^2,\\ y(3x+y)\leq\lambda^2,\ x\geq y\geq 1\end{array}\right\}$$

is

$$\frac{\lambda^2}{3}\log\left(\sqrt{2\lambda^2+1}+1\right)-\frac{\lambda^2}{6}(4\log 2+1)+\frac{\lambda}{2}-\frac{1}{6}\left(\sqrt{2\lambda^2+1}-1\right),$$

one can easily find positive constants C_1 and C_2 with

$$\left|a(\lambda;1)-\frac{\lambda^2}{3}\log\lambda\right|<C_1\lambda^2+C_2$$

for every positive λ. Therefore we get

$$\begin{aligned}
&\left|\frac{a(\lambda)}{\lambda^2\log\lambda}-\frac{1}{3}\sum_{k=1}^{[\lambda/2]}\frac{\mu(k)}{k^2}\right| \\
&=\left|\sum_{k=1}^{[\lambda/2]}\frac{\mu(k)}{k^2}\left\{\frac{a(\lambda/k;1)}{(\lambda/k)^2\log\lambda}-\frac{1}{3}\right\}\right| \\
&\le\sum_{k=1}^{[\lambda/2]}\frac{1}{k^2}\left\{\frac{\log k}{3\log\lambda}\right. \\
&\qquad\left.+\frac{\log(\lambda/k)}{\log\lambda}\left|\frac{a(\lambda/k;1)}{(\lambda/k)^2\log(\lambda/k)}-\frac{1}{3}\right|\right\} \\
&\le\sum_{k=1}^{[\lambda/2]}\left\{\frac{C_1}{k^2\log\lambda}+\frac{C_2}{\lambda^2\log\lambda}+\frac{\log k}{3k^2\log\lambda}\right\} \\
&<\frac{2C_1}{\log\lambda}+\frac{C_2}{2\lambda\log\lambda}+\frac{2\log\lambda+\lambda}{3\lambda\log\lambda}\longrightarrow 0\ (\lambda\to\infty).
\end{aligned}$$

We hence obtain

$$\begin{aligned}
\lim_{\lambda\to\infty}\frac{a(\lambda)}{\lambda^2\log\lambda}&=\frac{1}{3}\sum_{k=1}^{\infty}\frac{\mu(k)}{k^2} \\
&=\frac{1}{3}\prod_{\ell}\left(1-\frac{1}{\ell^2}\right)=\frac{1}{3\zeta(2)}=\frac{2}{\pi^2},
\end{aligned}$$

where in the product ℓ runs over all positive prime integers and ζ denotes the Riemann zeta function.

In order to estimate the cardinality of the set S, for a positive number λ we denote by $a^{\mathrm{o}}(\lambda)$ and by $a^{\mathrm{e}}(\lambda)$ the cardinality of the sets

$$\left\{(p,q)\in\mathbf{Z}\times\mathbf{Z}\;\middle|\;\begin{array}{l}p\text{ and }q\text{ are relatively prime}\\ \text{integers which satisfy}\\ pq\text{ is odd, }9p^2-q^2>2\lambda^2,\\ q(3p+q)<\lambda^2\text{ and }p>q\ge 1\end{array}\right\},$$

$$\left\{(p,q)\in\mathbf{Z}\times\mathbf{Z}\;\middle|\;\begin{array}{l}p\text{ and }q\text{ are relatively prime}\\ \text{integers which satisfy}\\ pq\text{ is even, }9p^2-q^2>2\lambda^2,\\ q(3p+q)<\lambda^2\text{ and }p>q\ge 1\end{array}\right\},$$

respectively. For a positive real number λ and a positive integer k we consider the following three sets

$$\left\{(P,Q)\in k\mathbf{Z}\times k\mathbf{Z}\;\middle|\;\begin{array}{l}P\text{ and }Q\text{ are positive}\\ \text{odd integers which satisfy}\\ P>Q,\ 9P^2-Q^2>2\lambda^2,\\ Q(3P+Q)<\lambda^2\end{array}\right\},$$

$$\left\{(p,q)\in\mathbf{Z}\times\mathbf{Z}\;\middle|\;\begin{array}{l}p+1>q\ge 1,\\ 9(p+1)^2-q^2>2\lambda^2,\\ q\{3(p+1)+q\}<\lambda^2\end{array}\right\},$$

$$\left\{(p,q)\in\mathbf{Z}\times\mathbf{Z}\;\middle|\;\begin{array}{l}p>q+1\ge 2,\\ 9p^2-(q+1)^2>2\lambda^2,\\ q\{3p+(q+1)\}<\lambda^2\end{array}\right\},$$

which are denoted by $A^{\mathrm{o}}(\lambda;k)$, $B_x(\lambda)$ and $B_y(\lambda)$ respectively, and put $a^{\mathrm{o}}(\lambda;k)$ the cardinality of the set $A^{\mathrm{o}}(\lambda;k)$. Since the correspondences $(2p+1,2q+1)\mapsto(p,q)$ of $A^{\mathrm{o}}(\lambda;1)$ to $B_x(\lambda/2)$ and $(p,q)\mapsto(2p+1,2q+1)$ of $B_y(\lambda/2)$ to $A^{\mathrm{o}}(\lambda;1)$ are injective, by applying the same argument on $a(\lambda;1)$ to the cardinalities of $B_x(\lambda)$ and $B_y(\lambda)$, we find positive constants C_1' and C_2' with

$$\left|a^{\mathrm{o}}(\lambda;1)-\frac{\lambda^2}{12}\log\lambda\right|<C_1'\lambda^2+C_2'$$

for every λ. As we have $a^{\mathrm{o}}(\lambda;k)=0$ for even k, we obtain

$$a^{\mathrm{o}}(\lambda)=\sum_{k=1}^{\infty}\mu(k)a^{\mathrm{o}}(\lambda;k)=\sum_{\substack{1\le k\le[\lambda/2]\\ k\text{ is odd}}}\mu(k)a^{\mathrm{o}}\left(\frac{\lambda}{k};1\right).$$

We therefore get along the same lines on $a(\lambda)$ that

$$\begin{aligned}
\lim_{\lambda\to\infty}\frac{a^{\mathrm{o}}(\lambda)}{\lambda^2\log\lambda}&=\frac{1}{12}\sum_{\substack{1\le k<\infty\\ k\text{ is odd}}}\frac{\mu(k)}{k^2}=\frac{1}{12}\prod_{\ell\,(\ne 2)}\left(1-\frac{1}{\ell^2}\right) \\
&=\frac{1}{12}\times\frac{4}{3}\times\frac{1}{\zeta(2)}=\frac{2}{3\pi^2}.
\end{aligned}$$

As $a(\lambda)=a^{\mathrm{o}}(\lambda)+a^{\mathrm{e}}(\lambda)$, we find

$$\lim_{\lambda\to\infty}\frac{a^{\mathrm{e}}(\lambda)}{\lambda^2\log\lambda}=\frac{4}{3\pi^2}.$$

Coming back to our situation, as we see for $\lambda\ge 4\pi/\sqrt{c}$ that

$$m_{\mathbf{C}P^n(c)}(\lambda)=a^{\mathrm{o}}\left(\frac{3\sqrt{2c}}{4\pi}\lambda\right)+a^{\mathrm{e}}\left(\frac{3\sqrt{2c}}{8\pi}\lambda\right),$$

we obtain

$$\begin{aligned}
&\lim_{\lambda\to\infty}\frac{m_{\mathbf{C}P^n(c)}(\lambda)}{\lambda^2\log\lambda} \\
&\quad=\frac{2}{3\pi^2}\times\frac{9c}{8\pi^2}+\frac{4}{3\pi^2}\times\frac{9c}{32\pi^2}=\frac{9c}{8\pi^4}.
\end{aligned}$$

□

For continuity of the function $m_{\mathbf{C}P^n(c)}$, we have the following.

(1) Generally, it is not left continuous at points $(2\pi/3)\sqrt{2q(3p+q)/c}$ for some relatively prime positive integers p, q with odd pq and $(4\pi/3)\sqrt{2q(3p+q)/c}$ for some relatively prime positive integers p, q with even pq.

(2) Generally, it is not right continuous at points $(2\pi/3)\sqrt{(9p^2-q^2)/c}$ for some relatively prime positive integers p, q with odd pq and $(4\pi/3)\sqrt{(9p^2-q^2)/c}$ for some relatively prime positive integers p, q with even pq.

(3) These points forms a discrete unbounded set.

(4) Except these points it is continuous.

Next we study the asymptotic behaviour of the multiplicity of length spectrum of circles on a complex hyperbolic space.

Theorem 2. *For a complex hyperbolic space* $\mathbf{C}H^n(-c)$ $(n \geq 2)$ *of constant holomorphic sectional curvature* $-c$, *the multiplicity* $m_{\mathbf{C}H^n(-c)}$ *of length spectrum of circles satisfies*

$$\lim_{\lambda\to\infty} \frac{m_{\mathbf{C}H^n(-c)}(\lambda)}{\lambda^2 \log\lambda} = \frac{9c}{8\pi^4}.$$

Proof. Following [A] the length spectrum of circles on $\mathbf{C}H^n(-c)$ is as follows.

$$\begin{aligned}\mathrm{LSpec}(\mathbf{C}H^n(-c)) &= (0,\infty)\cup(0,\infty)\\ &\cup\bigcup\left\{ I_{\tau(p,q)} \,\middle|\, \begin{array}{l} p \text{ and } q \text{ are relatively}\\ \text{prime positive integers}\\ \text{with } p>q \end{array}\right\},\end{aligned}$$

where the interval $I_{\tau(p,q)}$ is given by

$$I_{\tau(p,q)} = \begin{cases} \left(\dfrac{4\pi}{3\sqrt{c}}\sqrt{2q(3p-q)}, \infty\right), & \text{if } pq \text{ is even},\\ \left(\dfrac{2\pi}{3\sqrt{c}}\sqrt{2q(3p-q)}, \infty\right), & \text{if } pq \text{ is odd}, \end{cases}$$

and $m_{\mathbf{C}H^n(-c)}(\lambda) - 2$ hence coincides with the number of sets $I_{\tau(p,q)}$ containing λ. Therefore what we have to do is to study the cardinality of the set

$$\begin{aligned} S = &\left\{ (p,q) \,\middle|\, \begin{array}{l} p \text{ and } q \text{ are relatively prime pos-}\\ \text{itive integers which satisfy}\\ p>q,\ pq \text{ is even},\\ (4\pi/3)\sqrt{2q(3p-q)/c} < \lambda \end{array}\right\}\\ &\bigcup \left\{ (p,q) \,\middle|\, \begin{array}{l} p \text{ and } q \text{ are relatively prime pos-}\\ \text{itive integers which satisfy}\\ p>q,\ pq \text{ is odd},\\ (2\pi/3)\sqrt{2q(3p-q)/c} < \lambda \end{array}\right\}. \end{aligned}$$

Since the area of the set

$$\left\{(x,y)\in\mathbf{R}^2 \mid y(3x-y)\leq\lambda^2,\ x\geq y\geq 1\right\}$$

is $(\lambda^2/3)\log\lambda - (\lambda^2/6)(1+\log 2) + 1/3$, we obtain our result by just the same way as in the proof of Theorem 1. $\square$

2. The number of closed circles of prescribed geodesic curvature. In this section, we classify circles by their geodesic curvature and investigate the asymptotic behaviour of the number of congruency classes of closed circles of prescribed geodesic curvature. For a Riemannian manifold M we denote by $n_M(\lambda;\kappa)$ the number of congruency classes of closed circles of geodesic curvature κ on M with length not greater than λ. When M is a compact manifold of negative curvature, the asymptotic behaviour of the number of closed geodesics is well known: $\lim_{\lambda\to\infty} \lambda e^{-h_M\lambda} n_M(\lambda;0) = h_M$, where h_M is the topological entropy of the geodesic flow on the unit tangent bundle of M. Since for space forms the length spectrum of geodesics is trivial, we here consider the number of closed circles of prescribed geodesic curvature.

Let $\mathrm{Cir}_\kappa(M)$ denote the set of all congruency classes of circles with geodesic curvature κ on M. We set $\mathrm{LSpec}_\kappa(M) = \mathcal{L}_M(\mathrm{Cir}_\kappa(M))\cap\mathbf{R}$. For real space forms, since circles on these spaces are congruent if and only if they have the same geodesic curvature, length spectrum of circles of geodesic curvature κ consists of a single point:

$$\begin{aligned} &\mathrm{LSpec}_\kappa(S^n(c)) = \left\{2\pi/\sqrt{\kappa^2+c}\right\},\\ &\mathrm{LSpec}_\kappa(\mathbf{R}^n) = \{2\pi/\kappa\},\\ &\mathrm{LSpec}_\kappa(H^n(-c)) = \left\{2\pi/\sqrt{\kappa^2-c}\right\}, \quad \text{if } \kappa>\sqrt{c},\\ &\mathrm{LSpec}_\kappa(H^n(-c)) = \emptyset, \quad \text{if } \kappa\leq\sqrt{c}. \end{aligned}$$

For non-flat complex space forms we showed the following in [A] and [AM] when $n\geq 2$.

1) $\mathrm{LSpec}_\kappa(\mathbf{C}P^n(c))$ and $\mathrm{LSpec}_\kappa(\mathbf{C}H^n(-c))$ are discrete unbounded sets.

2) The multiplicities $m_{\mathbf{C}P^n(c)}(\lambda;\kappa)$, $m_{\mathbf{C}H^n(-c)}(\lambda;\kappa)$ of length spectrum of circles of geodesic curvature κ and of length λ on a complex projective space and a complex hyperbolic space satisfy

$$\begin{aligned}&\limsup_{\lambda\to\infty} m_{\mathbf{C}P^n(c)}(\lambda;\kappa)\\ &\qquad = \limsup_{\lambda\to\infty} m_{\mathbf{C}H^n(-c)}(\lambda;\kappa) = \infty.\end{aligned}$$

More precisely, the length spectrum of circles of geodesic curvature κ on a complex hyperbolic space is as follows (see [A]). When $\sqrt{c}/2 < \kappa \le \sqrt{c}$,

$$\mathrm{LSpec}_\kappa(\mathbf{C}H^n(-c)) = \left\{4\pi/\sqrt{4\kappa^2-c}\right\}$$
$$\bigcup\left\{4\pi\sqrt{\frac{3p^2+q^2}{3(4\kappa^2-c)}}\;\middle|\;\begin{array}{l}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which}\\ \text{satisfy } p>q \text{ and } pq \text{ is even}\end{array}\right\}$$
$$\bigcup\left\{2\pi\sqrt{\frac{3p^2+q^2}{3(4\kappa^2-c)}}\;\middle|\;\begin{array}{l}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which}\\ \text{satisfy } p>q \text{ and } pq \text{ is odd}\end{array}\right\}$$

and when $\kappa > \sqrt{c}$,

$$\mathrm{LSpec}_\kappa(\mathbf{C}H^n(-c)) = \left\{4\pi/\sqrt{4\kappa^2-c},\, 2\pi/\sqrt{\kappa^2-c}\right\}$$
$$\bigcup\left\{4\pi\sqrt{\frac{3p^2+q^2}{3(4\kappa^2-c)}}\;\middle|\;\begin{array}{l}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which}\\ \text{satisfy } p>\alpha_\kappa q \text{ and } pq \text{ is}\\ \text{even}\end{array}\right\}$$
$$\bigcup\left\{2\pi\sqrt{\frac{3p^2+q^2}{3(4\kappa^2-c)}}\;\middle|\;\begin{array}{l}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which}\\ \text{satisfy } p>\alpha_\kappa q \text{ and } pq \text{ is}\\ \text{odd}\end{array}\right\}.$$

Here α_κ (>1) denotes the number with

$$\frac{3\sqrt{3}c\kappa}{(4\kappa^2-c)^{3/2}} = \frac{9\alpha_\kappa^2-1}{(3\alpha_\kappa^2+1)^{3/2}}, \tag{2.1}$$

which is

i) monotone increasing,
ii) $\lim_{\kappa\to\infty}\alpha_\kappa = \infty$, $\lim_{\kappa\downarrow\sqrt{c}}\alpha_\kappa = 1$.

These expressions yield the following.

Theorem 3. *For a complex hyperbolic space $\mathbf{C}H^n(-c)$ of constant holomorphic sectional curvature $-c$ and complex dimension $n(\ge 2)$, we have*

$$\lim_{\lambda\to\infty}\lambda^{-2}n_{\mathbf{C}H^n(-c)}(\lambda;\kappa) = \begin{cases}\dfrac{\sqrt{3}}{16\pi^3}(4\kappa^2-c), & \sqrt{c}/2<\kappa\le\sqrt{c},\\[2mm] \dfrac{3\sqrt{3}(4\kappa^2-c)}{8\pi^4}\tan^{-1}(1/(\sqrt{3}\alpha_\kappa)), & \kappa>\sqrt{c}.\end{cases}$$

Proof. What we have to do is to study the cardinality $b^{\mathrm{o}}_\alpha(\lambda)$ of the set

$$\left\{(p,q)\in\mathbf{Z}\times\mathbf{Z}\;\middle|\;\begin{array}{l}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which satisfy}\\ pq \text{ is odd, } 3p^2+q^2\le\lambda^2 \text{ and}\\ p>\alpha q>0\end{array}\right\}$$

and the cardinality $b^{\mathrm{e}}_\alpha(\lambda)$ of the set

$$\left\{(p,q)\in\mathbf{Z}\times\mathbf{Z}\;\middle|\;\begin{array}{l}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which satisfy}\\ pq \text{ is even, } 3p^2+q^2\le\lambda^2 \text{ and}\\ p>\alpha q>0\end{array}\right\},$$

for $\alpha\ge 1$ and $\lambda>0$. First we study the cardinalities $b_\alpha(\lambda)$, $b_\alpha(\lambda;1)$ of the following sets

$$\left\{(p,q)\in\mathbf{Z}\times\mathbf{Z}\;\middle|\;\begin{array}{l}p \text{ and } q \text{ are relatively prime}\\ \text{integers with } 3p^2+q^2\le\lambda^2\\ \text{and } p>\alpha q>0\end{array}\right\},$$
$$\{(p,q)\in\mathbf{Z}\times\mathbf{Z}\mid 3p^2+q^2\le\lambda^2,\ p>\alpha q>0\}.$$

Put $C_\alpha = (1/(2\sqrt{3}))\tan^{-1}(1/(\sqrt{3}\alpha))$. As the area of the set $\{(x,y)\in\mathbf{R}^2 \mid 3x^2+y^2\le\lambda^2,\ x\ge\alpha y\ge 0\}$ is $C_\alpha\lambda^2$, we find positive constants c_1, c_2 with $|b_\alpha(\lambda;1)-C_\alpha\lambda^2|\le c_1\lambda+c_2$ for every positive λ. By a similar argument as in the proof of Theorem 1, one can easily obtain $\lim_{\lambda\to\infty}b_\alpha(\lambda)/\lambda^2 = 6C_\alpha/\pi^2$. Next we consider the following three sets for $\alpha\ge 1$:

$$\left\{(P,Q)\in\mathbf{Z}\times\mathbf{Z}\;\middle|\;\begin{array}{l}P \text{ and } Q \text{ are positive odd}\\ \text{integers which satisfy}\\ P>\alpha Q,\ 3P^2+Q^2\le\lambda^2\end{array}\right\},$$
$$\left\{(p,q)\in\mathbf{Z}\times\mathbf{Z}\;\middle|\;\begin{array}{l}3(p+1)^2+q^2\le\lambda^2,\\ p+1>\alpha q\ge 0\end{array}\right\},$$
$$\left\{(p,q)\in\mathbf{Z}\times\mathbf{Z}\;\middle|\;\begin{array}{l}3(p+1)^2+(q+1)^2\le\lambda^2,\\ p+1>\alpha(q+1),\ q\ge 1\end{array}\right\}.$$

We denote these sets by $B^{\mathrm{o}}_\alpha(\lambda;1)$, $U_\alpha(\lambda)$ and $V_\alpha(\lambda)$ respectively. Since the correspondences $B^{\mathrm{o}}_\alpha(\lambda;1)\ni(2p+1,2q+1)\mapsto(p,q)\in U(\lambda/2)$ and $V(\lambda/2)\ni(p,q)\mapsto(2p+1,2q+1)\in B^{\mathrm{o}}_\alpha(\lambda;1)$ are injective, by the same way as in the proof of Theorem 1, we obtain

$$\lim_{\lambda\to\infty}b^{\mathrm{o}}_\alpha(\lambda)/\lambda^2 = 2C_\alpha/\pi^2,\quad \lim_{\lambda\to\infty}b^{\mathrm{e}}_\alpha(\lambda)/\lambda^2 = 4C_\alpha/\pi^2.$$

Setting $\alpha_\kappa = 1$ for $\sqrt{c}/2<\kappa\le\sqrt{c}$, we get

$$\begin{aligned}&\lim_{\lambda\to\infty}\lambda^{-2}n_{\mathbf{C}H^n(-c)}(\lambda;\kappa)\\ &= \lim_{\lambda\to\infty}\frac{1}{\lambda^2}\left\{b^{\mathrm{o}}_{\alpha_\kappa}\left(\frac{\sqrt{3(4\kappa^2-c)}}{2\pi}\lambda\right) + b^{\mathrm{e}}_{\alpha_\kappa}\left(\frac{\sqrt{3(4\kappa^2-c)}}{4\pi}\lambda\right)\right\}\\ &= \frac{3(4\kappa^2-c)}{4\pi^2}\times\frac{2C_{\alpha_\kappa}}{\pi^2} + \frac{3(4\kappa^2-c)}{16\pi^2}\times\frac{4C_{\alpha_\kappa}}{\pi^2}\\ &= \frac{3\sqrt{3}}{8\pi^4}(4\kappa^2-c)\tan^{-1}(1/(\sqrt{3}\alpha_\kappa)).\end{aligned}$$

□

We here study the dependence of

$$\mathcal{N}_{\mathbf{C}H^n(-c)}(\kappa) = \lim_{\lambda\to\infty} \lambda^{-2} n_{\mathbf{C}H^n(-c)}(\lambda;\kappa)$$

on κ.

Proposition 4. *For a complex hyperbolic space* $\mathbf{C}H^n(-c)$ $(n \geq 2)$ *the function* $\mathcal{N}_{\mathbf{C}H^n(-c)}(\kappa)$ *is monotone increasing for* $\sqrt{c}/2 < \kappa \leq \sqrt{c}$ *and monotone decreasing for* $\kappa \geq \sqrt{c}$. *It takes the maximum value* $3\sqrt{3}c\pi^{-3}/16$ *at* $\kappa = \sqrt{c}$. *It satisfies*

$$\lim_{\kappa\downarrow\sqrt{c}/2} \mathcal{N}_{\mathbf{C}H^n(-c)}(\kappa) = 0,\ \lim_{\kappa\to\infty} \mathcal{N}_{\mathbf{C}H^n(-c)}(\kappa) = \frac{9c}{16\pi^4}.$$

Proof. By using (2.1) we see that

$$\lim_{\kappa\to\infty} \frac{\kappa^2}{\alpha_\kappa} = \lim_{\kappa\to\infty} \frac{\kappa^2}{\alpha_\kappa} \times \frac{3\sqrt{3}c\kappa}{(4\kappa^2-c)^{3/2}} \times \frac{(3\alpha_\kappa^2+1)^{3/2}}{9\alpha_\kappa^2-1}$$
$$= \frac{3\sqrt{3}c}{8} \times \frac{1}{\sqrt{3}} = \frac{3c}{8}.$$

Thus for the behaviour at infinity we have

$$\lim_{\kappa\to\infty} \mathcal{N}_{\mathbf{C}H^n(-c)}(\kappa)$$
$$= \frac{3\sqrt{3}}{8\pi^4} \lim_{\kappa\to\infty} \frac{4\kappa^2-c}{\sqrt{3}\alpha_\kappa} \times \sqrt{3}\alpha_\kappa \tan^{-1}(1/(\sqrt{3}\alpha_\kappa))$$
$$= \frac{3\sqrt{3}}{8\pi^4} \times \frac{3c}{2\sqrt{3}} = \frac{9c}{16\pi^4}.$$

It is trivial that $\mathcal{N}_{\mathbf{C}H^n(-c)}(\kappa)$ is monotone increasing for $\sqrt{c}/2 < \kappa < \sqrt{c}$, so we study on $\kappa > \sqrt{c}$. Differentiating both sides of (2.1) we have

$$\frac{\alpha_\kappa(\alpha_\kappa^2-1)}{(3\alpha_\kappa^2+1)(9\alpha_\kappa^2-1)}\alpha_\kappa' = \frac{8\kappa^2+c}{27\kappa(4\kappa^2-c)}.$$

Since $s \mapsto (1/s)\tan^{-1} s$ is monotone decreasing for $s > 0$, we find

$$\frac{d}{d\kappa}\mathcal{N}_{\mathbf{C}H^n(-c)}(\kappa)$$
$$= 8\kappa\tan^{-1}(1/(\sqrt{3}\alpha_\kappa)) - \sqrt{3}(4\kappa^2-c)\frac{\alpha_\kappa'}{3\alpha_\kappa^2+1}$$
$$< \frac{8\kappa}{\sqrt{3}\alpha_\kappa} - \frac{\sqrt{3}(8\kappa^2+c)(9\alpha_\kappa^2-1)}{27\kappa\alpha_\kappa(\alpha_\kappa^2-1)}$$
$$= -\frac{64\kappa^2+c(9\alpha_\kappa^2-1)}{9\sqrt{3}\kappa\alpha_\kappa(\alpha_\kappa^2-1)} < 0,$$

hence we obtain $\mathcal{N}_{\mathbf{C}H^n(-c)}(\kappa)$ is monotone decreasing for $\kappa > \sqrt{c}$. □

The length spectrum of circles of geodesic curvature κ on a complex projective space is as follows (cf. [AM]).

$$\mathrm{LSpec}_\kappa(\mathbf{C}P^n(c)) = \left\{2\pi/\sqrt{\kappa^2+c},\, 4\pi/\sqrt{4\kappa^2+c}\right\}$$
$$\bigcup\left\{4\pi\sqrt{\frac{3p^2+q^2}{3(4\kappa^2+c)}}\ \middle|\ \begin{array}{l} p \text{ and } q \text{ are relatively prime} \\ \text{integers which satisfy} \\ pq \text{ is even and } p > \beta_\kappa q > 0 \end{array}\right\}$$
$$\bigcup\left\{2\pi\sqrt{\frac{3p^2+q^2}{3(4\kappa^2+c)}}\ \middle|\ \begin{array}{l} p \text{ and } q \text{ are relatively prime} \\ \text{integers which satisfy} \\ pq \text{ is odd and } p > \beta_\kappa q > 0 \end{array}\right\}.$$

Here β_κ (≥ 1) denotes the number with

$$\frac{3\sqrt{3}c\kappa}{(4\kappa^2+c)^{3/2}} = \frac{9\beta_\kappa^2-1}{(3\beta_\kappa^2+1)^{3/2}},$$

which satisfies

i) $\beta_{\sqrt{2c}/4} = 1$,

ii) monotone decreasing when $0 < \kappa \leq \sqrt{2c}/4$, and monotone increasing when $\kappa \geq \sqrt{2c}/4$,

iii) $\lim_{\kappa\downarrow 0}\beta_\kappa = \lim_{\kappa\to\infty}\beta_\kappa = \infty$.

Since the situation is almost the same as for a complex hyperbolic space we obtain the following.

Theorem 5. *For a complex projective space* $\mathbf{C}P^n(c)$ $(n \geq 2)$ *of constant holomorphic sectional curvature* c*, we have for* $\kappa > 0$

$$\mathcal{N}_{\mathbf{C}P^n(c)}(\kappa) = \frac{3\sqrt{3}(4\kappa^2+c)}{8\pi^4}\tan^{-1}(1/(\sqrt{3}\beta_\kappa)).$$

In particular, $\mathcal{N}_{\mathbf{C}P^n(c)}(\sqrt{2c}/4) = 3\sqrt{3}c\pi^{-3}/32$. *The function* $\mathcal{N}_{\mathbf{C}P^n(c)}(\kappa)$ *is monotone increasing for* $0 < \kappa \leq \sqrt{2c}/4$ *and satisfies*

$$\lim_{\kappa\downarrow 0}\mathcal{N}_{\mathbf{C}P^n(c)}(\kappa) = 0 \text{ and } \lim_{\kappa\to\infty}\mathcal{N}_{\mathbf{C}P^n(c)}(\kappa) = \frac{9c}{16\pi^4}.$$

It is not clear for the author on the behaviour of $\mathcal{N}_{\mathbf{C}P^n(c)}(\kappa)$ for $\kappa > \sqrt{2c}/4$. At least, it is decreasing for $\sqrt{2c}/4 < \kappa < \sqrt{2c}/4 + \epsilon$.

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