The fundamental group of the moduli space of polygons in the Euclidean plane

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1. Introduction. Let $\mathcal{C}_n$ be the configuration space of planar polygons with n vertices, each edge having length 1 in $\mathbf{R}^2$;

$$\mathcal{C}_n = \{(u_1, \ldots, u_n) \mid |u_{i+1} - u_i| = 1 \ (1 \le i \le n-1), \\ |u_1 - u_n| = 1\} \subset (\mathbf{R}^2)^n.$$

Note that $\mathrm{Iso}(\mathbf{R}^2)$, the isometry group of $\mathbf{R}^2$, naturally acts on $\mathcal{C}_n$. We define

$$M_n = \mathcal{C}_n / \operatorname{Iso}^+(\mathbf{R}^2)$$
$$M'_n = \mathcal{C}_n / \operatorname{Iso}(\mathbf{R}^2)$$

where $\mathrm{Iso}^+(\mathbf{R}^2)$ is the orientation preserving isometry group. Identifying $\mathbf{R}^2$ with $\mathbf{C}$, we can write M_n as

$$M_n = \{(z_1, \ldots, z_{n-1}) \mid z_1 + \ldots + z_{n-1} - 1 = 0\} \subset (S^1)^{n-1}$$

and $M'_n = M_n/\sigma$ where σ is the complex conjugation. Note that the action of σ on M_n is free if n is odd and has fixed points if n is even.

For $n \le 5$, the explicit topological type of M_n is known ([1], [2], [4], and [8]).

It is known that M_{2m+1} is a smooth manifold, while M_{2m} is a manifold with singular points ([5], [7], and [9]).

The purpose of this paper is to study the fundamental group of M_{2m} and M'_{2m}.

In [7], Y. Kamiyama and M. Tezuka showed by the Morse theory that if n is odd, the inclusion

$$i_n : M_n \to (S^1)^{n-1}$$

is a homotopy equivalence up to certain dimension.

Theorem 1.1 ([7]).

$$(i_{2m+1})_* : \pi_q(M_{2m+1}) \to \pi_q((S^1)^{2m})$$

is an isomorphism for $q \le m - 2$.

T.Hinokuma and H.Shiga showed the corresponding result for even n using the other Morse function ([3]). We give the alternative proof based on the method of [7] and show the following.

Theorem 1.2.

$$(i_{2m})_* : \pi_q(M_{2m}) \to \pi_q((S^1)^{2m-1})$$

is an isomorphism for $q \le m - 2$.

Remark 1.3. In [3], much more information about the topology of M_n is obtained.

In particular we have the following.

Corollary 1.4. $\pi_1(M_{2m})$ *is abelian for* $m \ge 3$.

The topology of M'_n is studied in [6] and he determined the fundamental group of M'_{2m+1} as well as almost all the homology groups of M'_n.

We determine the fundamental group of M'_{2m}.

Theorem 1.5. *For* $m \ge 3$

$$\pi_1(M'_{2m}) = \mathbf{Z}/2.$$

2. Outline of the proof of Theorem 1.2. Following [7], we consider the function

$$g_{2m-1} : (S^1)^{2m-1} \to \mathbf{R}$$

defined by $g_{2m-1}(z_1, \ldots, z_{2m-1}) = |z_1 + \ldots + z_{2m-1} - 1|^2$. Note that $g^{-1}_{2m-1}(0) = M_{2m}$ is a "critical singular submanifold".

Proposition 2.1 ([7]).

$$(z_1, \ldots, z_{2m-1}) \in (S^1)^{2m-1} - M_{2m}$$

is a critical point of g_{2m-1} *if and only if* $z_i = \pm 1$ $(1 \le i \le 2m-1)$. *Moreover such points are non-degenerate with index greater than or equal to* m.

Proposition 2.2. *There exist* $0 < \varepsilon < 2$ *and a retraction* $r : g^{-1}_{2m-1}([0, \varepsilon]) \to M_{2m}$.

Combining these propositions, we see that $(i_{2m})_* : \pi_q(M_{2m}) \to \pi_q((S^1)^{2m-1})$ is injective for $q \le m-2$. By [7], we know that $(i_{2m})_* : H_1(M_{2m}; \mathbf{Z}) \to H_1((S^1)^{2m-1}; \mathbf{Z})$ is an isomorphism, which complete the proof.

3. Outline of the proof of Theorem 1.5. We set

$$\Sigma_{n-1} = \{(z_1, \ldots, z_{n-1}) \mid z_i = \pm 1\} \subset (S^1)^{n-1}$$

$$\Sigma^1_{n-1} = \{(z_1, \ldots, z_{n-1}) | z_i = \pm 1, \sum z_i = 1\} \subset \Sigma_{n-1}$$

$$\Sigma^2_{n-1}=\{(z_1,\dots,z_{n-1})|z_i=\pm 1, \sum z_i \neq 1\}\subset\Sigma_{n-1}.$$

Note that Σ_{n-1} is the fixed point set of the action of σ, the complex conjugation, on $(S^1)^{n-1}$ and $\Sigma^1_{n-1} = \Sigma_{n-1} \cap M_n$. We define V_{n-1} and V'_{n-1} by

$$V_{n-1} = (S^1)^{n-1} - \Sigma_{n-1}/\sigma$$
$$V'_{n-1} = (S^1)^{n-1} - \Sigma^2_{n-1}/\sigma$$

respectively. Then we have the following map of covering spaces.

$$\begin{tikzcd}
\mathbf{Z}/2 \arrow[r, equal] \arrow[d] & \mathbf{Z}/2 \arrow[d] \\
M_n - \Sigma^1_{n-1} \arrow[r] \arrow[d] & (S^1)^{n-1} - \Sigma_{n-1} \arrow[d] \\
M_n - \Sigma^1_{n-1}/\sigma \arrow[r] & V_{n-1}.
\end{tikzcd}$$

Let $i'_n\colon M'_n \to V'_{n-1}$ be the inclusion. Since $\Sigma^1_{2m} = \emptyset$ and $V_{2m} = V'_{2m}$, we have the following ([6]).

Theorem 3.1 ([6])**.**

$$(i'_{2m+1})_*\colon \pi_q(M'_{2m+1}) \to \pi_q(V'_{2m})$$

are isomorphisms for $q \leq m-2$ and an epimorphism for $q = m-1$.

For even n, we have the following.

Lemma 3.2. *The inclusion induces an isomorphism of fundamental group $\pi_1(M_{2m}-\Sigma^1_{2m-1}) \cong \pi_1(V_{2m-1})$ for $m \geq 4$.*

By Van Kampen Theorem and diagram chasing with a little more work for $m = 3$, we have the following lemmas which complete the proof of Theorem 1.5.

Lemma 3.3.

$$(i'_{2m})_*\colon \pi_1(M'_{2m}) \to \pi_1(V'_{2m-1})$$

is an isomorphism for $m \geq 3$.

Lemma 3.4. $\pi_1(V'_{2m-1}) = \mathbf{Z}/2$.

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