"Hasse principle" for $SL_n(D)$

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1. Notation and results. Extending the usage of language in Galois cohomology, T. Ono defined "Hasse principle" for any group G (cf. [2]). We know that the Hasse principle holds for $G=$ abelian, dihedral, quatenion, $PSL_2(\mathbf{Z})$, $PSL_2(\mathbf{F}_p)$ (cf. [2]), free groups ([3]), symmetric groups and alternating groups ([4]).
Let D be an Euclidean domain(for examples $D=\mathbf{Z}$, $D=\mathbf{F}_p$). Put $\varepsilon=(-1)^{n-1}$ and we define in $SL_n(D)$

$$S=\begin{pmatrix} 0 & \dots & \dots & 0 & \varepsilon \\ 1 & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix},$$

$$T_\mu=\begin{pmatrix} 1 & \mu & 0 & \dots & 0 \\ 0 & \ddots & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix},\ T=T_1.$$

Then $SL_n(\mathbf{Z})$ is generated by S and T (cf.[1]). Similarly using Eucledian algorithm we can prove that $SL_n(D)$ is generated by S and $\{T_\mu \mid \mu\in D\}$. Using this fact, T. Ono proved that $SL_2(D)$ enjoys the Hasse principle (unpublished). In this paper, we shall prove more generally the following

Theorem. *For any natural number n, $SL_n(D)$ and $PSL_n(D)$ enjoy the Hasse principle.*

M. Mazur [5] noticed that $f(x)$ is a cocycle iff $g(x)=f(x)x$ is an endomorphism of G, $f(x)$ is a local coboundary iff $g(x)\sim x$ (conjugate in G) for each $x\in G$ and $f(x)$ is a global coboundary iff $g(x)$ is an inner automorphism. Therefore "Hasse principle" is equivalent to say that "any endomorphism of G which satisfies $g(x)\sim x$ for each $x\in G$ must be an inner automorphism".

2. Proof of the theorem. Let $g(x)$ be an endomorphism which satisfies $g(x)\sim x$ for each $x\in G$. We may assume $g(S)=S$, $g(T)=M^{-1}TM$ where

$$M=\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & & \vdots \\ g_1 & g_2 & \dots & g_n \end{pmatrix}=\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \vdots \\ \boldsymbol{g} \end{pmatrix},$$

$$M^{-1}=\begin{pmatrix} x_1 & \\ x_2 & \\ \vdots & * \\ x_n & \end{pmatrix}.$$

Then

$$g(T)=E+M^{-1}E_{12}M=$$
$$\begin{pmatrix} 1+x_1b_1 & x_1b_2 & \dots & x_1b_n \\ x_2b_1 & 1+x_2b_2 & \dots & x_2b_n \\ \vdots & \vdots & & \vdots \\ x_nb_1 & x_nb_2 & \dots & 1+x_nb_n \end{pmatrix}.$$

where E_{ij} is the matrix unit whose ij-element is 1 and the other elements are 0. Put

$$\tilde{M}=\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{b} \\ \boldsymbol{v}_3 \\ \boldsymbol{v}_4 \\ \vdots \\ \boldsymbol{v}_n \end{pmatrix}= \tag{1}$$

$$\begin{pmatrix} b_2 & b_3 & b_4 & \dots & b_n & \varepsilon b_1 \\ b_1 & b_2 & b_3 & \dots & b_{n-1} & b_n \\ \varepsilon b_n & b_1 & b_2 & \dots & b_{n-2} & b_{n-1} \\ \varepsilon b_{n-1} & \varepsilon b_n & b_1 & \dots & b_{n-3} & b_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \varepsilon b_3 & \varepsilon b_4 & \varepsilon b_5 & \dots & b_1 & b_2 \end{pmatrix}.$$

Then after a little calculation we have

$$|xE-Sg(T)|=$$
$$x^n-\sum_{i=1}^{n-1}(x_1b_{i+1}+x_2b_{i+2}+\cdots+x_{n-i}b_n$$
$$+\varepsilon x_{n-i+1}b_1+\cdots+\varepsilon x_nb_i)x^{n-i}-\varepsilon=$$

$$x^n - \begin{vmatrix} \boldsymbol{v}_1 \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \vdots \\ \boldsymbol{g} \end{vmatrix} x^{n-1} - \begin{vmatrix} \varepsilon\boldsymbol{v}_n \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \vdots \\ \boldsymbol{g} \end{vmatrix} x^{n-2} - \cdots - \begin{vmatrix} \varepsilon\boldsymbol{v}_3 \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \vdots \\ \boldsymbol{g} \end{vmatrix} x - \varepsilon,$$

$$|xE - ST| = x^n - x^{n-1} - \varepsilon.$$

As $g(x) \sim x$, $Sg(T)$ and ST are conjugate in $SL_n(D)$. Therefore their characteristic polynomials are equal. So we have

$$\begin{vmatrix} \boldsymbol{v}_1 \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \vdots \\ \boldsymbol{g} \end{vmatrix} = 1, \begin{vmatrix} \varepsilon\boldsymbol{v}_n \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \vdots \\ \boldsymbol{g} \end{vmatrix} = 0, \ldots, \begin{vmatrix} \varepsilon\boldsymbol{v}_3 \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \vdots \\ \boldsymbol{g} \end{vmatrix} = 0. \tag{2}$$

As $g(T)$ is only depend on $\boldsymbol{b}, \boldsymbol{c}, \ldots, \boldsymbol{g}$, we may assume $\boldsymbol{a} = \boldsymbol{v}_1$. Let ζ be an n-th root of ε taken from the algebraic closure of the quotient field of D. Then ζ is an algebraic integer over D. Using (2)

$$(3)\ \zeta = \begin{vmatrix} \boldsymbol{b} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \vdots \\ \boldsymbol{g} \end{vmatrix} + \zeta \begin{vmatrix} \boldsymbol{v}_1 \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \vdots \\ \boldsymbol{g} \end{vmatrix} + \zeta^2 \begin{vmatrix} \varepsilon\boldsymbol{v}_n \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \vdots \\ \boldsymbol{g} \end{vmatrix} + \cdots + \zeta^{n-1} \begin{vmatrix} \varepsilon\boldsymbol{v}_3 \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \vdots \\ \boldsymbol{g} \end{vmatrix}$$

$$= (b_1 + \zeta b_2 + \cdots + \zeta^{n-1} b_n) \begin{vmatrix} 1\ \zeta^{-1} \ldots\ \zeta^{-(n-1)} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \vdots \\ \boldsymbol{g} \end{vmatrix},$$

$$\prod_{\zeta^n=\varepsilon} \zeta = 1, \quad |\tilde{M}| = \prod_{\zeta^n=\varepsilon} (b_1 + \zeta b_2 + \cdots + \zeta^{n-1} b_n).$$

As D is integrally closed, we get

$$|\tilde{M}| = unit. \tag{4}$$

From (1), (2), (4) we have $\dim\langle \boldsymbol{v}_1, \boldsymbol{b}, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_n\rangle = \dim\langle \boldsymbol{v}_1, \boldsymbol{b}, \boldsymbol{c}, \ldots, \boldsymbol{g}\rangle = n$ and $\dim\langle \boldsymbol{v}_i, \boldsymbol{b}, \boldsymbol{c}, \ldots, \boldsymbol{g}\rangle = n-1$ $(3 \le i \le n)$. So we have $\langle \boldsymbol{b}, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_n\rangle = \langle \boldsymbol{b}, \boldsymbol{c}, \ldots, \boldsymbol{g}\rangle$. Therefore the first column of $\tilde{M}^{-1}$ and M^{-1} are equal and $g(T) = M^{-1}TM = \tilde{M}^{-1}T\tilde{M}$. As $g(S) = S = \tilde{M}^{-1}S\tilde{M}$ we have

$$A \in \langle S, T\rangle \Longrightarrow g(A) = \tilde{M}^{-1}A\tilde{M}. \tag{5}$$

Next we shall prove that $|\tilde{M}| = \beta^n$ for some unit β in D. In $SL_n(\mathbf{Z})$, we have $\langle S, T\rangle = SL_n(\mathbf{Z})$. Using natural homomorphism from $\mathbf{Z}$ to D we have in $SL_n(D)$

$$K = \begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix} \in \langle S, T\rangle. \tag{6}$$

From (5) and $g(x) \sim x$, we have

$$g(K) = \tilde{M}^{-1}K\tilde{M} = N^{-1}KN \quad {}^{\exists}N \in SL_n(D)$$

$$K(\tilde{M}N^{-1}) = (\tilde{M}N^{-1})K. \tag{7}$$

Comparing the elements of both sides of (7), we have

$$\tilde{M}N^{-1} = \begin{pmatrix} \beta & & * \\ & \ddots & \\ 0 & & \beta \end{pmatrix} \quad {}^{\exists}\beta \in D.$$

Therefore we have $\beta^n = |\tilde{M}| =$ unit. So β must be a unit in D. Put $M_1 = \beta^{-1}\tilde{M}$. Then $M_1 \in SL_n(D)$, $g(T) = M_1^{-1}TM_1$, $SM_1 = M_1S$. Therefore from now on we may assume $g(S) = S$, $g(T) = T$.

Next we treat T_μ instead of T. In the same way as for T we can find $\tilde{M}_\mu$ such that $g(T_\mu) = \tilde{M}_\mu^{-1}T_\mu\tilde{M}_\mu$, $S\tilde{M}_\mu = \tilde{M}_\mu S$. Let $\mathcal{S}$ be the subset of $\langle S, T\rangle$ defined by

$$\mathcal{S} = \{E + E_{1j} | 2 \le j \le n\} \cup \{E + E_{i2} | 3 \le i \le n\}.$$

For any $A \in \mathcal{S}$, we have $AT_\mu = T_\mu A$. As $g(A) = A$, we have

$$(8)\ g(AT_\mu) = Ag(T_\mu) = g(T_\mu A) = g(T_\mu)A \quad {}^{\forall}A \in \mathcal{S}$$

Comparing the elements of (8), we have

$$g(T_\mu) = \begin{pmatrix} b & c & & 0 \\ & b & & \\ & & \ddots & \\ 0 & & & b \end{pmatrix} \quad {}^{\exists}b, {}^{\exists}c \in D.$$

Hence, $\tilde{M}_\mu$ must be a scalar matrix. So $g(T_\mu) = T_\mu$. As $SL_n(D)$ is generated by S and T_μ, $SL_n(D)$ enjoys the Hasse principle.

When n is even, $PSL_n(D) = SL_n(D)/\pm E$. From $Sg(T) \sim ST$ (in $PSL_n(D)$) we may have $SM^{-1}TM \sim -ST$ (in $SL_n(D)$). In this case we may assume $\boldsymbol{a} = -\boldsymbol{v}_1$ and instead of $\tilde{M}$ we use $\tilde{L}$ whose first, second, …, n-th rows are $-\boldsymbol{v}_1$, $\boldsymbol{b}$, $-\boldsymbol{v}_3$, $\boldsymbol{v}_4$, …, $-\boldsymbol{v}_{n-1}$, $\boldsymbol{v}_n$. Then we have $M^{-1}TM = \tilde{L}^{-1}T\tilde{L}$ and $S = -\tilde{L}^{-1}S\tilde{L}$. Instead of (7) we may have $K(\tilde{L}N^{-1}) = -(\tilde{L}N^{-1})K$. But from this equation we have $\tilde{L}N^{-1} = 0$, a contradiction. From (8) we may have $A(\tilde{M}_\mu^{-1}T_\mu\tilde{M}_\mu) = -(\tilde{M}_\mu^{-1}T_\mu\tilde{M}_\mu)A$ for some $A \in \mathcal{S}$. But from this equation we have $\tilde{M}_\mu^{-1} T_\mu \tilde{M}_\mu = 0$, a contradiction. If $\tilde{M}_\mu S = -S\tilde{M}_\mu$, we have $g(T_\mu) = T_\mu^{-1}$. In the same way we have $g(T_{1+\mu}) = T_{1+\mu}$ or $T_{1+\mu}^{-1}$. But $g(T_{1+\mu}) = g(TT_\mu) = Tg(T_\mu)$. So $g(T_\mu)$ must be T_μ. Therefore $PSL_n(D)$ enjoys the Hasse principle.

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References

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