Uniqueness of the nonlinear term of a boundary value problem from the first bifurcating branch

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1. Statement of results. In this paper we shall consider the inverse problem of determining the nonlinear term g of the boundary value problem

$$(1.1)\quad \begin{cases} u'' + [\lambda - q(x)]u = g(u), \ 0 < x < 1, \\ u'(0) = u(1) = 0, \end{cases}$$

from its first bifurcating branch. From a viewpoint of physical applications, the investigation of the inverse problem can be regarded as a study to determine unknown inhomogeneity of elastic materials such as springs or rubbers by searching a modulus of elasticity which matches given period of vibration for each amplitude. Related inverse problems have been studied by Denisov [2], Lorenzi [8], Denisov and Lorenzi [3], Kamimura [7], which can be considered as investigations to determine inhomogeneity by measuring the dependence of the initial velocity and the displacement at a fixed time on the modulus of elasticity. There have been few investigations concerning inverse problems of determining unknown nonlinear terms in nonlinear differential equations from some measured date for their solutions, outside of these works.

An existence result for the inverse problem mentioned in the beginning was established by Iwasaki and Kamimura [4]. The purpose of the present paper is to establish a uniqueness result for the problem.

Let q be a real function of class $C[0, 1]$ and assume that g is a real function of class $C^1(\boldsymbol{R})$ satisfying $g(0) = g'(0) = 0$. As a representation of the first bifurcating branch of (1.1) in $\boldsymbol{R}^2$, let $\Gamma(g)$ be the set of $(\lambda, h) \in \boldsymbol{R}^2$ for which there exists a solution $u(x)$ of (1.1) such that (i) $u(x) \neq 0$ for any $x \in [0, 1)$; (ii) $u(0) = h$. The assumption $g(0) = g'(0) = 0$ implies that the linearized problem of (1.1) at the trivial solution $u(x) \equiv 0$ is :

$$(1.2)\quad \begin{cases} u'' + [\lambda - q(x)]u = 0, \ 0 < x < 1, \\ u'(0) = u(1) = 0. \end{cases}$$

Therefore the set $\Gamma(g)$ bifurcates at the point $(\lambda_1, 0)$ from the trivial solution $u(x) \equiv 0$, where λ_1 is the first eigenvalue of the problem (1.2) (see [4,§2], also see [1,9,10] for general theory).

Throughout the paper, we assume that the first eigenfunction $v_1(x)$ of (1.2) satisfies the following three conditions:

(A1) $v_1''(0) < 0$.
(A2) $v_1'(x) < 0$ for $0 < x \leq 1$.
(A3) $v_1''(x)v_1(x) \leq 2v_1'(x)^2$ for $0 \leq x < 1$.

This is an assumption on q. It should be pointed-out that if $\max_{0 \leq x \leq 1} q(x) < \lambda_1$ then (A1)–(A3) hold. For other sufficient conditions for (A1)–(A3) the reader may refer to [4, Remark 4.8].

We use the following two function spaces. Let $0 < \alpha < 1/2$ and let X, Y be function spaces defined by

$$X := \Big\{ g(h) \in C^1(\boldsymbol{R}) \,|\, g(0) = g'(0) = 0, \ \sup_{h,k \in R, h \neq k} \frac{|(1 + |k|^\alpha)g'(k) - (1 + |h|^\alpha)g'(h)|}{|k - h|^\alpha} < \infty \Big\},$$

$$Y := \Big\{ \lambda(h) \in C(\boldsymbol{R}) | h\lambda'(h) \in C(\boldsymbol{R}), \ \lambda(0) = \lambda_1, \ \sup_{h \in R} |\lambda(h)| + \sup_{h,k \in R, h \neq k} \frac{||k|^{3/2}(1 + |k|^\alpha)\lambda'(k) - |h|^{3/2}(1 + |h|^\alpha)\lambda'(h)|}{|k - h|^{\alpha + 1/2}} < \infty \Big\}.$$

For $\lambda(h) \in Y$, let $u(h, x)$ denote the solution of the initial value problem

$$(1.3)\quad \begin{cases} u'' + [\lambda(h) - q(x)]u = g(u), \\ u(0) = h, \ u'(0) = 0. \end{cases}$$

Clearly if $(\lambda(h), h) \in \Gamma(g)$ then $u(h, 1) = 0$.

The main result of the present paper is stated as follows :

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Theorem 1.1. *Let $q(x)$ be a real function of class $C^2[0, 1]$ and suppose that the conditions* (A1) − (A3) *hold. Suppose that, for $\lambda(h) \in Y$, there e ists a function $g \in X$ satisfying*

(1.4) $\quad \Gamma(g) = \{(\lambda(h), h) \mid h \in \boldsymbol{R}\setminus\{0\}\},$

and that, for each $h \in \boldsymbol{R}\setminus\{0\}$, the solution $u(h, x)$ of (1.3) *satisfies the following three conditions*:

(i) $u''(h, 0) < 0$,

(ii) $u'(h, x) \neq 0$ *for any* $x \in (0, 1]$,

(iii) *the solution $w(h, x)$ of the initial value problem*

$$\begin{cases} w'' + [\lambda(h) - q(x) - g'(u(h, x))]w = 0, \\ w(0) = 0,\ w'(0) = 1, \end{cases}$$

satisfies $w(h, 1) \neq 0$.

Then g satisfying (1.4) *is unique in X.*

In [4] it was proved that if $\lambda(h)$ is sufficiently near the straight line $\lambda_0(h) \equiv \lambda_1$ in the norm of Y then there exists a function $g \in X$ satisfying (1.4), and that the correspondence $\lambda(h) \longmapsto g(h)$ is continuous as a mapping from X to Y. Moreover, as is easily seen, the conditions (i) − (iii) are satisfied if $\sup_{h\in R} |\lambda(h) - \lambda_1| + \sup_{h\in R} |g'(h)|$ is sufficiently small. Hence, as a direct consequence of Theorem 1.1, we have:

Theorem 1.2. *Let $q(x)$ be a real function of class $C^2[0, 1]$ and suppose that the conditions* (A1) − (A3) *hold. If $\lambda(h) \in Y$ is sufficiently near the straight line $\lambda_0(h) \equiv \lambda_1$ in the norm of Y then, for $g_1, g_2 \in X$,*

$$\Gamma(g_1) = \Gamma(g_2) = \{(\lambda(h), h) \mid h \in \boldsymbol{R}\setminus\{0\}\} \Rightarrow g_1 = g_2.$$

This result implies that the first bifurcating curve of (1.1) controls its nonlinear term g. In particular we have:

Corollary 1.3. *Under the same assumption on q as in Theorem* 1.2,

$$\Gamma(g) = \{(\lambda_1, h) \mid h \in \boldsymbol{R}\setminus\{0\}\},\ g \in X \Rightarrow g = 0.$$

Remark 1.4. For the second bifurcating branch, the conclusion in Corollary 1.3 is not true (see [6]).

In the case $q(x) \equiv 0$, we easily get $u'(h, x) w'(h, x) - u''(h, x)w(h, x) \equiv 0$, from which it follows that the condition (iii) is satisfied for $q(x) \equiv 0$. Furthermore, as is readily checked (see [5,§2]), other conditions (i) and (ii) are also satisfied. Hence we have:

Corollary 1.5. *Let $q(x) \equiv 0$. Then, for $g_1, g_2 \in X$,*

$$\Gamma(g_1) = \Gamma(g_2) \Rightarrow g_1 = g_2.$$

In the next section we shall present an outline of the proof of Theorem 1.1. More detailed proof will be published elsewhere. Our approach consists in combining the idea of [4] with the techniques developed in [7] to trace the proof of the implicit function theorem (see e.g. [10, §2.7.2]).

2. Sketch of the proof. Let $g_0 \in X$ be a function satisfying (1.4) and conditions (i) − (iii) of Theorem 1.1. Moreover let $g_1 \in X$ be a function satisfying (1.4). We use the notation $\tilde{g}(h) = g_1(h) - g_0(h)$ and put, for $0 \le \sigma \le 1$, $g_\sigma(h) = g_0(h) + \sigma\tilde{g}(h)$. Our goal is to show that $\tilde{g}(h) \equiv 0$.

Let $v(h, x; \sigma)$ be the solution of

$$\begin{cases} v'' + [\lambda(h) - q(x)]v = h^{-1} g_\sigma(hv), \\ v(0) = 1,\ v'(0) = 0. \end{cases}$$

In the case $h = 0$ we define $v(0, x; \sigma) = v_1(x)$, where $v_1(x)$ is the first eigenfunction normalized by $v_1(0) = 1$. By means of the assumptions $\sup_{h\in R} | h^{-1}g_\sigma(h) | < \infty$, $\sup_{h\in R} | \lambda(h) | < \infty$, it follows that $v(h, x; \sigma)$ can be defined for any $h \in \boldsymbol{R}$, $0 \le x \le 1$, $0 \le \sigma \le 1$. It is clear from the assumption

$$\Gamma(g_0) = \Gamma(g_1) = \{(\lambda(h), h) \mid h \in \boldsymbol{R}\setminus\{0\}\}$$

that

(2.1) $\quad v(h, 1; 0) = v(h, 1; 1) = 0.$

The theory of dependence of solutions on parameters shows that $v(h, x; \sigma)$ is differentiable in σ and the derivative $v_\sigma(h, x; \sigma)$ satisfies the following:

$$\text{(2.2)}\quad \begin{cases} v''_\sigma + [\lambda(h) - q(x) - g'_\sigma(hv(h, x; \sigma))]v_\sigma \\ \qquad\qquad = h^{-1}\tilde{g}(hv(h, x; \sigma)), \\ v_\sigma(0) = v'_\sigma(0) = 0. \end{cases}$$

Let $w_1(h, x; \sigma)$, $w_2(h, x; \sigma)$ are solutions of the equation

$$w'' + [\lambda(h) - q(x) - g'_\sigma(hv(h, x; \sigma))]w = 0$$

satisfying the conditions $w_i^{(j-1)}(h, 0; \sigma) = \delta_{ij}$, $i, j = 1, 2$, and put

$$\text{(2.3)}\quad G(h, x, t; \sigma) := w_1(h, t; \sigma)w_2(h, x; \sigma) - w_2(h, t; \sigma)w_1(h, x; \sigma).$$

Then (2.2) is solved as

$$v_\sigma(h, x; \sigma) = h^{-1}\int_0^x G(h, x, t; \sigma)\tilde{g}(hv(h, t; \sigma))dt.$$

This, together with (2.1), yields

$$\text{(2.4)}\quad 0 = \int_0^1 v_\sigma(h, 1; \sigma)d\sigma$$

$$= h^{-1}\int_0^1 d\sigma \int_0^1 G(h,\,1,\,t\,;\,\sigma)\tilde{g}(hv(h,\,t\,;\,\sigma))dt.$$

We define an operator T by

(2.5) $$Tg(h):\;= h^{-1}\int_0^1 G(h,\,1,\,t\,;\,0)g(hv(h,\,t\,;\,0))dt.$$

Moreover we set

$$\phi(h):=-h^{-1}\int_0^1 d\sigma\int_0^1\{G(h,1,t;\sigma)\tilde{g}(hv(h,t;\sigma)) - G(h,1,t;0)\tilde{g}(hv(h,t;0))\}dt. \tag{2.6}$$

Then (2.4) can be written as

$$T\tilde{g} = \phi. \tag{2.7}$$

We now introduce two function spaces as follows:

$$X(H):=\Big\{g(h)\in C^1[-H,H]\,|\,g(0)=g'(0)=0,\quad \|g\|_{X(H)}:=\sup_{h,k\in[-H,H],h\neq k}\frac{|g'(k)-g'(h)|}{|k-h|^\alpha}<\infty\Big\},$$

$$Y(H):=\Big\{\phi(h)\in C[-H,H]\,|\,h\phi'(h)\in C[-H,H],\,\phi(0)=0,\quad \|\phi\|_{Y(H)}:=\sup_{h,k\in[-H,H],h\neq k}\frac{\big||k|^{3/2}\phi'(k)-|h|^{3/2}\phi'(h)\big|}{|k-h|^{\alpha+1/2}}<\infty\Big\},$$

where $H>0$. Moreover we use the notation

$$|g|_H:=\sup_{h\in[-H,H]}|g'(h)|.$$

Concerning the function $\phi\,(h)$ defined in (2.6), we have the following estimate:

Lemma 2.1. *Let $H_1>0$. If $v'(h,x;\sigma)\neq 0$ for $|h|\le H_1$, $0<x\le 1$, $0\le\sigma\le 1$ then, for each $H\le H_1$, $\phi(h)$ belongs to the space $Y(H)$ and the norm is estimated as*

$$\|\phi\|_{Y(H)}\le M\|\tilde{g}\|_{X(H)}\,|\tilde{g}|_H^\alpha,$$

where M is independent of H.

To show that if H is suitably small then the operator T defined in (2.5) is an isomorphism of $X(H)$ onto $Y(H)$, we approximate T by an operator L defined by

$$Lg(h):=h^{-1}\int_0^1 G(0,1,t\,;0)g(hv_1(t))dt$$

By denoting the residual $T-L$ by R we obtain the decomposition

$$T=L+R.$$

It follows from (2.3) that $G\;(0,1,t\,;0\;)=-v_1'(1)^{-1}v_1(t)$, which yields

$$Lg(h)=-h^{-1}v_1'(1)^{-1}\int_0^1 v_1(t)g(hv_1(t))dt.$$

This operator was studied in [4, §4]. The following is a direct consequence of [4, Theorem 4.7]. (Here we have used the assumptions (A1)—(A3).)

Lemma 2.2. *For any $H>0$, the operator L is an isomorphism from $X(H)$ onto $Y(H)$. Moreover the operator norm $\|L^{-1}\|_H$ is uniformly bounded, that is, $\|L^{-1}\|_H\le M$ with some constant M independent of H.*

Furthermore a tedious estimation to

$$Rg(h)=h^{-1}\int_0^1\{G(h,\,1,\,t\,;\,0)g(hv(h,\,t\,;\,0)) - G(0,\,1,\,t\,;\,0)g(hv_1(t))\}dt$$

yields the following:

Lemma 2.3. *For sufficiently small $H>0$, the operator R is a bounded operator from $X(H)$ to $Y(H)$. The operator norm $\|R\|_H$ converges to 0 as $H\to 0$, that is, $\lim_{H\to 0}\|R\|_H=0$.*

By Lemmas 2.2 and 2.3 we have $\|L^{-1}R\|_H\le\frac{1}{2}$ for sufficiently small $H>0$. In this case the equation (2.7) can be solved as $\tilde{g}=(I+L^{-1}R)^{-1}L^{-1}\phi$ by the Neumann series, where I denotes the identity operator. This, together with Lemma 2.1, leads to

$$\|\tilde{g}\|_{X(H)}\leqq 2M\|\phi\|_{Y(H)}\le M'\|\tilde{g}\|_{X(H)}\,|\tilde{g}|_H^\alpha$$

for small $H>0$, where M' is independent of H. But, in view of $\tilde{g}'(0)=0$, if $H>0$ is sufficiently small then $M'|\tilde{g}|_H^\alpha<1$. This shows that $\|\tilde{g}\|_{X(H)}=0$. Thus we have proved the following:

Lemma 2.4. *There exists $H>0$ such that $\tilde{g}(h)=0$ for any $h\in[-H,\,H]$.*

We now define a number H^* by

$$H^*:=\sup\{H\,|\,\tilde{g}(h)=0\text{ for any }h\in[-H,H]\}. \tag{2.8}$$

By Lemma 2.4 we have $H^*>0$. Assume that $H^*<\infty$. Then, by the conditions (i), (ii) and continuity of $v(h,\,x\,;\,\sigma)$ it follows that there exists $H_1>H^*$ such that

$$v'(h,\,x\,;\,\sigma)\neq 0\text{ for }|h|\le H_1,\;0<x\le 1,\;0\le\sigma\le 1.$$

Let $v^{-1}\,(h,\,y\,)$ denote the inverse function of $v\,(h,x\,;\,0\,)$. Then, by the definition (2.8), the equation (2.7) can be rewritten in the form

$$\int_{\pm H^*}^h\frac{W(h,\,s)}{(h^2-s^2)^{1/2}}s\tilde{g}(s)ds=h^2\phi(h),\;H^*\le\pm h\le H_1,$$

for $\pm\,h>0$, respectively. Here we set

$$W(h,\,s):=\frac{G(h,\,1,\,v^{-1}(h,\,s/h)\,;\,0)}{-\,v'(h,\,v^{-1}(h,\,s/h)\,;\,0)}\frac{(h^2-s^2)^{1/2}}{s}.$$

An elementary calculation shows that

$$W(h, h) = \frac{G(h, 1, 0; 0)}{(-v''(h, 0; 0))^{1/2}},$$

where $-v''(h, 0; 0) > 0$ by the condition (ii). From (2.3) and the condition (iii) we have

$$G(h, 1, 0; 0) = w_2(h, 1; 0) = w(h, 1) \neq 0.$$

Therefore, by a standard method (see [11, §41]) of reduction to Volterra integral equations of the second kind, we can prove the following:

Lemma 2.5. *For each* $H \in [H^*, H_1]$,

$$\| \tilde{g} \|_{X(H)} \leq C \| \psi \|_{Y(H)},$$

where C *is a constant independent of* H.

Combining Lemmas 2.1 and 2.5 we obtain

$$\| \tilde{g} \|_{X(H)} \leq CM \| \tilde{g} \|_{X(H)} | \tilde{g} |_H^{\alpha},$$

from which it follows that there exists $H_- > H_*$ such that $\tilde{g}(h) = 0$ for any $h \in [-H_+, H_+]$. This contradicts the fact H^* was the largest of such numbers. Therefore we conclude that $H^* = \infty$, which implies $\tilde{g}(h) = 0$ for any $h \in \boldsymbol{R}$.

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