Construction of Jacobi cusp forms

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1. Introduction. In [2; Theorem 3. 1], certain linear operators on spaces of Jacobi forms were introduced for cusp forms of one variable. These operators are represented as adjoint operators to "product operators" with respect to the Petersson inner product. The purpose of the present paper is to extend this result to the case of general "Jacobi cusp forms" in place of cusp forms of one variable (see Theorem 3.1 below). This extension is obtained in the same way as in [2], but the L-series which appear in our Theorem are of a different type from those in the theorem in [2].

2. Jacobi forms and Petersson inner product. For the theory of Jacobi forms we refer to [3]. We write $\Gamma_1 = SL_2(\boldsymbol{Z})$ for the full modular group and $\mathfrak{H}$ for the upper half-plane. The elements $\gamma = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu))$ of the Jacobi group $\Gamma_1^J = \Gamma_1 \ltimes \boldsymbol{Z}^2$ operates on $\mathfrak{H} \times \boldsymbol{C}$ in the usual way by

$$\gamma \circ (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right)$$

and for given positive integers k and m on functions $\phi : \mathfrak{H} \times \boldsymbol{C} \to \boldsymbol{C}$ by

$$\phi|_{k,m}\gamma = (c\tau + d)^{-k}$$
$$\times \exp(-2\pi i m(\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} - \lambda^2\tau - 2\lambda z))$$
$$\times \phi(\gamma \circ (\tau, z)).$$

Let $J_{k,m}$ be the space of Jacobi forms of weight k and index m, i.e. the space of holomorphic functions $\phi : \mathfrak{H} \times \boldsymbol{C} \to \boldsymbol{C}$ satisfying $\phi|_{k,m}\gamma = \phi$ for all $\gamma \in \Gamma_1^J$ and let the Fourier expansion of ϕ be

$$\phi(\tau, z) = \sum_{n, r \in Z, r^2 \leq 4mn} c(n, r) e^{2\pi i(n\tau + rz)}.$$

We write $J_{k,m}^{\text{cusp}}$ for the subspace of cusp forms of $J_{k,m}$, which require $c(n, r) = 0$ unless $r^2 < 4mn$. For $\phi_1, \phi_2 \in J_{k,m}$ such that $\phi_1 \times \phi_2$ is cuspidal, we define the Petersson inner product by

$$< \phi_1, \phi_2 > = \int_{\Gamma_1^J \backslash \mathfrak{H} \times C} \phi_1(\tau, z) \overline{\phi_2(\tau, z)} v^k e^{-4\pi m y^2/v} dV_J,$$

where $\tau = u + iv$, $z = x + iy$ and $dV_J = v^{-3}$ $dudvdxdy$ is an invariant measure under the action of Γ_1^J on $\mathfrak{H} \times \boldsymbol{C}$. The space $(J_{k,m}^{\text{cusp}}, <, >)$ is a finite dimensional Hilbert space. The following lemma will be used later.

Lemma 2.1. *Let ϕ be a function in $J_{k,m}$ with Fourier coefficients $c(n, r)$ and put the discriminant $D := r^2 - 4mn$. Then $c(n, r)$ depend only on D and on the residue class of r modulo $2m$. Furthermore, if $k > 3$ and ϕ is a cusp form, then*

$$c(n, r) \ll |D|^{k/2 - 1/2} (D < 0).$$

Remark 2.1. *If we have only the condition $k > 3$, then*

$$c(n, r) \ll |D|^{k - 3/2} (D < 0).$$

For a proof of the first statement, see [3, pp. 22-23], and for the second statement (the estimates of Fourier coefficients), see [1, pp 308]. Hereafter we shall write simply $c_\mu(D')$ for $c(n, r)$ where $D' = |D| = 4mn - r^2$ and $\mu \equiv r$ (*mod* $2m$).

3. Construction of Jacobi forms. First we remind the definition of the Jacobi Poincaré series.

Definition 3.1. *For $n, r \in \boldsymbol{Z}$ with $r^2 < 4mn$ we denote by*

$$P_{k,m;n,r}(\tau, z) = \sum_{\gamma \in \Gamma_{1,\infty}^J \backslash \Gamma_1^J} e^{2\pi i(n\tau + rz)}|k, m\gamma$$

the (n, r)-th Jacobi Poincaré series of weight k and index m. (*Note that the group*

$$\Gamma_{1,\infty}^J = \left\{ \left[\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, (0, \mu) \right] \middle| t, \mu \in \boldsymbol{Z} \right\}$$

is a stabilizer of $e^{2\pi i(n\tau + rz)}$ in Γ_1^J.)

It is well-known that $P_{k,m;n,r}(\tau, z) \in J_{k,m}^{\text{cusp}}$ for weight $k > 2$ (see [4]). This (infinite) series has the following property, and it is expanded in a neighborhood of cusp as follows.

Lemma 3.1. *Let $\phi(\tau, z) \in J_{k,m}^{\text{cusp}}$ with $\phi(\tau, z) = \sum_{n, r \in Z, r^2 < 4mn} c(n, r) e^{2\pi i(n\tau + rz)}$. Then*

$$< \phi(\tau, z), P_{k,m;n,r}(\tau, z) > = \alpha_{k,m} (4mn - r^2)^{3/2 - k} c(n, r),$$

where

$$\alpha_{k,m} = \frac{m^{k-2}\Gamma(k - 3/2)}{2\pi^{k-3/2}}.$$

We obtain the following lemma by using this lemma:

Lemma 3.2. *The Fourier expansion of $P_{k,m;n,r}(\tau, z)$ is given by*

$$P_{k,m;n,r}(\tau, z) = \sum_{n_1, r_1 \in Z, r_1^2 < 4mn_1} \gamma^{\pm}_{k,m;n,r}(n_1, r_1) e^{2\pi i(n_1\tau + r_1 z)},$$

where $\pm 1 = (-1)^k$, $\gamma^{\pm}_{k,m;n,r}(n_1, r_1)$ *is* $\gamma_{k,m;n,r}(n_1, r_1)$ *symmetrized or anti-symmetrized with respect to* r_1, *i. e.*, $\gamma^{\pm}_{k,m;n,r}(n_1, r_1) = \gamma_{k,m;n,r}(n_1, r_1) \pm \gamma_{k,m;n,r}(n_1, -r_1)$, *and*

$$\gamma_{k,m;n,r}(n_1, r_1) = \delta_m(n, r; n_1, r_1) + i^k\sqrt{2m}\,\pi m^{-1} (D_1/D)^{k/2-3/4} \times \sum_{c \ge 1} H_{m,c}(n, r; n_1, r_1) J_{k-3/2}\left(\frac{\pi}{mc} \times \sqrt{D_1 D}\right)$$

where $D_1 = r_1^2 - 4mn_1$, $D = r^2 - 4mn$,

$$\delta_m(n, r; n_1, r_1) = \begin{cases} 1 \text{ if } D_1 = D,\ r_1 \equiv r (\mathrm{mod}\, 2m) \\ 0 \text{ otherwise,} \end{cases}$$

$$H_{m,c}(n, r; n_1, r_1) = c^{-3/2} \sum_{\lambda (\mathrm{mod}\, c)} \sum_{\substack{\rho (\mathrm{mod}\, c) \\ (\rho, c) = 1}} \exp\left(2\pi i\left(\frac{(m\lambda^2 + r\lambda + n)\rho^{-1} + n_1\rho + r_1\lambda}{c}\right)\right) \times \exp\left(2\pi i\left(\frac{rr_1}{2mc}\right)\right),$$

and $J_{k-3/2}$ *is the Bessel function of order* $k - 3/2$.

Proofs of these lemmas are given in [4: pp 519-522]. From this lemma we can easily deduce Lemma 2.1 in the case of Jacobi-Poincaré series.

Now we shall construct Jacobi cusp forms by applying above Lemmas 2.1 and 3.1.

Theorem 3.1. *Suppose that* $k_1, k_2 \in \boldsymbol{N}$ *such that* $k_1 > 4$, $k_2 > 3$, *and* $m_1, m_2 \in \boldsymbol{N}$. *Let*

$$\phi_1(\tau, z) = \sum_{\substack{n_1, r_1 \in Z \\ 4(m_1+m_2)n_1 - r_1^2 > 0}} a(n_1, r_1) e^{2\pi i(n_1\tau + r_1 z)} \in J^{\mathrm{cusp}}_{k_1+k_2, m_1+m_2},$$

and

$$\phi_2(\tau, z) = \sum_{\substack{n_2, r_2 \in Z \\ 4m_2n_2 - r_2^2 > 0}} b(n_2, r_2) e^{2\pi i(n_2\tau + r_2 z)} \in J^{\mathrm{cusp}}_{k_2, m_2}.$$

Then

$$\Phi_{\phi_2}(\phi_1)(\tau, z) = \sum_{\substack{n, r \in Z \\ 4m_1 n - r^2 > 0}} c(n, r) e^{2\pi i(n\tau + rz)}$$

is a Jacobi cusp form of weight k_1 *and index* m_1, *where*

$$c(n, r) = \frac{(4m_1 n - r^2)^{k_1-3/2}(m_1+m_2)^{k_1+k_2-2}\Gamma(k_1+k_2-3/2)}{\pi^{k_2} m_1^{k_1-2}\Gamma(k_1 - 3/2)} \times \sum_{n_2 \ge 1} \sum_{\substack{r_2 \in Z \\ 4m_2n_2 > r_2^2 \\ 4(m_1(n+n_2)+m_2n) - r(r+2r_2) \ge 0}} \frac{a(n+n_2, r+r_2)\overline{b(n_2, r_2)}}{(4(m_1+m_2)(n+n_2) - (r+r_2)^2)^{k_1+k_2-3/2}}.$$

Remark 3.1. *The inner sum of the series is a finite sum. Therefore the series converges if* $k_1 > 4$, $k_2 > 3$ *as shown in the following way. From Lemma 2.1 the series is easily seen to be equal to*

$$\cdots \times \sum_{\mu (\mathrm{mod}\, 2m_2)} \sum_{\mu' (\mathrm{mod}\, 2(m_1+m_2))} \sum_{\substack{D \ge 1 \\ D \equiv -\mu'^2 (\mathrm{mod}\, 4(m_1+m_2))}} \frac{a_{\mu'}(D)}{D^{k_1+k_2-3/2}} \times \sum_{r_2} b_\mu\left(\frac{m_2(D - 4(m_1+m_2)n) + m_2 r(r+2r_2) - m_1 r_2^2}{m_1 + m_2}\right)$$

where r_2 *runs over* $\boldsymbol{Z}$ *such that* $r_2 \equiv \mu (\mathrm{mod}\, 2m_2)$, $r_2 \equiv \mu' - r (\mathrm{mod}\, 2(m_1+m_2))$, *and*

$$\frac{m_2}{m_1}\left(r - \sqrt{r^2 + \frac{m_1}{m_2}(D + r^2 - 4(m_1+m_2)n)}\right) < r_2 < \frac{m_2}{m_1}\left(r + \sqrt{r^2 + \frac{m_1}{m_2}(D + r^2 - 4(m_1+m_2)n)}\right).$$

Therefore we find that it converges if $k_1 + k_2 > k_2 + 4 > 4$, *i.e.*, $k_1 > 4$ *by virture of Lemma 2.1.*

Remark 3.2. *In case* $m_2 = 0$, *we assume that* $k_1 > 5$ *and* $\phi_2(\tau, z) = \sum_{n_2 \ge 1} b(n_2) e^{2\pi i n_2 \tau}$. *Then this theorem is the same as the one given in* [2: *Theorem 3.1*].

Remark 3.3. *If* $\phi_2(\tau, z) = P_{k_2, m_2}$, $c(n, r)$ *involves special value of certain Dirichlet series which involves Kloosterman-type sums* $H_{m,c}(n, r; n_1, r_1)$ *and Fourier coefficients of* $\phi_1(\tau, z)$ *by Lemma 3.2.*

Proof. Let us define

$$F(\tau, z) = \phi_1(\tau, z)\overline{\phi_2(\tau, z)} v^{k_2} e^{\frac{-4\pi m_2 y^2}{v}},$$

where $\tau = u + iv$, $z = x + iy$. We can easily check that $F(\tau, z)$ satisfies the transformation law of Jacobi forms, and has a reasonable speed of growth in $\mathfrak{H} \times \boldsymbol{C}$, so that the integral

$$(\phi, F) = \int_{\Gamma_1^J \backslash \mathfrak{H} \times \boldsymbol{C}} \phi(\tau, z)\overline{F(\tau, z)} v^{k_1} e^{-4\pi m_1 y^2/v} dV_J$$

is well defined as it converges for every $\phi \in J^{\mathrm{cusp}}_{k_1, m_1}$. Since the map on $J^{\mathrm{cusp}}_{k_1, m_1}: \phi \rightarrow (\phi, F)$ is linear, there exists a unique function $f(\tau, z) = \sum_{\substack{n, r \in Z \\ 4m_1 n - r_1^2 > 0}} c(n, r) e^{2\pi i(n\tau + rz)} \in J^{\mathrm{cusp}}_{k_1, m_1}$ satisfying $< \phi, f > = (\phi, F)$ for all ϕ in $J^{\mathrm{cusp}}_{k_1, m_1}$ by Riesz's theorem. (The above linear operator which maps F to f is a holomorphic projection of F.) Take the Poincaré series $P_{k_1, m_1; n, r}(\tau, z) \in J^{\mathrm{cusp}}_{k_1, m_1}$ for any cusp form ϕ such that $< \phi, f > = (\phi, F)$. Then, from the definition of two inner products and the Lemma 3.1 we get

$$c(n, r) = \frac{(4m_1 n - r^2)^{k_1-3/2}}{\alpha_{k_1,m_1}} < f(\tau, z), P_{k_1,m_1;n,r}(\tau, z) >$$

$$= \frac{(4m_1 n - r^2)^{k_1-3/2}}{\alpha_{k_1,m_1}} (F(\tau, z), P_{k_1,m_1;n,r}(\tau, z)).$$

By the Lebesgue dominated convergence theorem, we can interchange the sum and integral in $(F, P_{k_1,m_1;n,r})$, so that

$$(F(\tau, z), P_{k_1,m_1;n,r}(\tau, z))$$

$$= \int_{\Gamma_1^J\backslash\mathfrak{H}\times C} F(\tau, z)\overline{P_{k_1,m_1;n,r}(\tau, z)}v^{k_1}e^{-4\pi m_1 y^2/v}dV_J \tag{3.1}$$

$$= \sum_{\gamma\in\Gamma_{1,\infty}^J\backslash\Gamma_1^J}\int_{\Gamma_1^J\backslash\mathfrak{H}\times C} F(\tau, z)\overline{e^{2\pi i(n\tau+rz)}|_{k_1,m_1}\gamma}v^{k_1}e^{-4\pi m_1 y^2/v}dV_J \tag{3.2}$$

$$= \int_{\Gamma_{1,\infty}^J\backslash\mathfrak{H}\times C} F(\tau, z)\overline{e^{2\pi i(n\tau+rz)}}v^{k_1}e^{-4\pi m_1 y^2/v}dV_J, \tag{3.3}$$

where in the last line we have used the usual Rankin unfolding method. Substituting the Fourier expansions of ϕ_1 and ϕ_2 in the definition of F, we find the n_0-th coefficients $\beta(F ; n_0 ; v ; z)$ of F in the variable $e^{2\pi iu}$, is given by

$$\beta(F ; n_0 ; v ; z) = \begin{cases} \displaystyle\sum_{\substack{n_2,r_2\in Z\\ 4m_2n_2>r_2^2}}\ \sum_{\substack{r_1\in Z\\ 4(m_1+m_2)(n_0+n_2)>r_1^2}} a(n_0+n_2, r_1)\overline{b(n_2, r_2)}e^{-2\pi(2n_2+n_0)v} \\ \quad\times e^{-2\pi(r_1+r_2)y}e^{2\pi i(r_1-r_2)x}v^{k_2}e^{\frac{-4\pi m_2 y^2}{v}} \quad \text{if } n_0\ge 0 \\ \displaystyle\sum_{\substack{n_2,r_2\in Z\\ 4m_2(n_2-n_0)>r_2^2}}\ \sum_{\substack{r_1\in Z\\ 4(m_1+m_2)n_0>r_1^2}} a(n_2, r_1)\overline{b(n_2-n_0, r_2)}e^{-2\pi(2n_2-n_0)v} \\ \quad\times e^{-2\pi(r_1+r_2)y}e^{2\pi i(r_1-r_2)x}v^{k_2}e^{\frac{-4\pi m_2 y^2}{v}} \quad \text{if } n_0< 0. \end{cases}$$

Putting them into (3.3) and interchanging the sum and integral, we have

$$(3.3) = \sum_{n_0\in Z}\int_{\Gamma_{1,\infty}^J\backslash\mathfrak{H}\times C}\beta(F ; n_0 ; v ; z)e^{2\pi i(n_0-n)u}$$
$$\times e^{-2\pi(nv+ry)}e^{-2\pi irx}v^{k_1-3}e^{\frac{-4\pi m_1 y^2}{v}} dudvdxdy. \tag{3.4}$$

We also observe that a fundamental domain for the action of $\Gamma_{1,\infty}^J$ on $\mathfrak{H}\times C$ is $([0, 1])\times([0, \infty])\times([0, 1])\times\boldsymbol{R}$. And we know that $\int_0^1 e^{2\pi i(a-b)u}du = \delta_{a,b}$ where $\delta_{a,b}$ is Kronecker's delta. So, as $n\ge 1$, we obtain

$$(3.4) = \sum_{\substack{n_2,r_2\in Z\\ 4m_2n_2>r_2^2}}\ \sum_{\substack{r_1\in Z\\ 4(m_1+m_2)(n+n_2)>r_1^2}} a(n+n_2, r_1)\overline{b(n_2, r_2)}$$

$$\times\int_0^\infty\int_0^1\int_{-\infty}^\infty e^{-4\pi(n_2+n)v}e^{-2\pi(r_1+r_2+r)y}$$
$$\times e^{2\pi i(r_1-r_2-r)x}v^{k_1+k_2-3}e^{\frac{-4\pi(m_1+m_2)y^2}{v}} dvdxdy.$$

$$= \sum_{n_2\ge 1}\ \sum_{\substack{n_2,r_2\in Z\\ 4m_2n_2>r_2^2\\ 4(m_1(n+n_2)+m_2n)-r(r+2r_2)\ge 0}} a(n+n_2, r+r_2)\overline{b(n_2, r_2)}$$
$$\times\int_0^\infty e^{-4\pi(n_2+n)v}v^{k_1+k_2-3}$$
$$\times\int_{-\infty}^\infty e^{-4\pi((r_2+r)y+\frac{(m_1+m_2)y^2}{v})} dydv. \tag{3.5}$$

Since we have

$$\int_{-\infty}^\infty e^{-4\pi((r_2+r)y+\frac{(m_1+m_2)y^2}{v})} dy$$
$$= e^{\frac{\pi(r+r_2)^2v}{m_1+m_2}}\int_{-\infty}^\infty e^{\frac{-4\pi(m_1+m_2)}{v}(y+\frac{(r_1+r_2)v}{2(m_1+m_2)})^2} dy$$
$$= \frac{\sqrt{v}e^{\frac{\pi(r+r_2)^2v}{m_1+m_2}}}{2\sqrt{m_1+m_2}},$$

using the well-known Euler's expression of Gamma function $\Gamma(z)$ as an integral, we get

$$(3.5) = \frac{(m_1+m_2)^{k_1+k_2-2}\Gamma(k_1+k_2-3/2)}{2\pi^{k_1+k_2-3/2}}$$
$$\times\sum_{n_2\ge 1}\ \sum_{\substack{r_2\in Z\\ 4m_2n_2>r_2^2\\ 4(m_1(n+n_2)+m_2n)-r(r+2r_2)\ge 0}}$$
$$\frac{a(n+n_2, r+r_2)\overline{b(n_2, r_2)}}{(4(m_1+m_2)(n+n_2)-(r+r_2)^2)^{k_1+k_2-3/2}}.$$

Finally, we can prove the Theorem by taking f for $\Phi_{\phi_2}(\phi_1)$.

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