A generating function for rational curves on rational surfaces

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1. Introduction. In [4], Yamagishi constructed families of elliptic curves of Mordell-Weil rank at least five and with a nontrivial rational 2-torsion point parametrized by certain rational curves. Her method is to construct a kind of universal family of such elliptic curves whose parameter space is a (2,2,2)-complete intersection in $\boldsymbol{P}^5$ and to find appropriate rational curves in it using the fact that it has a structure of elliptic surface. On the other hand, in [1], Bremner considered the problem to find all the rational curves of low degrees on a specific $K3$ surface. In view of their results, it should be of some interest to consider the following problem:

Given a projective surface S and a positive integer d, can one determine the number of algebraic equivalence classes of rational curves of degree d as explicitly as possible?

In order to detect the behavior of the number as a function of d, it will be useful to consider its generating function (see Section 2 for its definition). One of the main results of this paper is that for a class of important rational surfaces like Del Pezzo surfaces, rational ruled surfaces, the corresponding generating functions can be computed explicitly and that they are always *rational functions in x.*

Since for these surfaces we know the structure of their Néron-Severi groups quite explicitly, we can translate the problem into the one to solve a family of Diophantine equations parametrized by the degree of rational curves (see Section 3, for example, for the typical case of equation we must consider). Details will appear elsewhere.

2. Main theorem. Let k denote an arbitrary algebraically closed field. In this section we define a kind of counting function of rational curves on projective surfaces over k, and formulate our main theorem about the function.

Let S denote a nonsingular projective surface defined over k, and let $NS(S)$ denote its Néron-Severi group. We fix a projective imbedding of S, and denote by $H \in NS(S)$ the class of its hyperplane section. We consider the set R_d of algebraic equivalence classes of irreducible curves of arithmetic genus zero with degree d (w.r.t. H) on S. Let r_d denote the number of elements in R_d. (We will see below these numbers are finite.) The main concern of the present article is the following generating function of these numbers:

Definition 2.1. *We denote by $Z_{rat}(S\,;x)$ the formal power series $1 + \sum_{d\geq 1} r_d x^d$, and call it* zeta function for rational curves on S.

Theorem 2.2. *The zeta function for rational curves $Z_{rat}(\boldsymbol{P}^2\,;x)$ of $\boldsymbol{P}^2$ is given by*

$$Z_{rat}(\boldsymbol{P}^2\,;x) = 1 + \sum_{d\geq 1} x^d = \frac{1}{1-x}.$$

Let X_e denote the rational ruled surface $\boldsymbol{P}(O_{\mathrm{p}^1} \oplus O_{\mathrm{p}^1}(-e))$. We fix a section C_0 of X_e with $C_0^2 = -e$. Let f denote the algebraic equivalence class of its fiber. Then we know that $NS(X_e)$ is generated by C_0 and f (see [2, Chapter V], for example).

Theorem 2.3. *Let X_e denote the rational ruled surface $\boldsymbol{P}(O_{\mathrm{p}^1} \oplus O_{\mathrm{p}^1}(-e))$, regarded as embedded into a projective space by means of the very ample divisor $C_0 + (e+1)f$ in the notation of* [2]. *Then the zeta function for rational curves $Z_{rat}(X_e\,;x)$ is given by*

$$Z_{rat}(X_e\,;x) = \begin{cases} \dfrac{1+x-x^2+x^3}{1-x}, & \text{if } e = 0, \\ \dfrac{1+x-x^2}{1-x}, & \text{if } e = 1, \\ \dfrac{1+x-2x^2+x^{e+1}}{1-x}, & \text{if } e \geq 2. \end{cases}$$

Remark. In general, the most general very ample divisor on X_e is given by $H_{m,n} = mC_0 + (e+n)f$, $m, n > 0$. If we regard X_e as embedded in a projective space by means of the linear system $|H_{m,n}|$, then we can also show that the associated zeta function is rational for any $m, n > 0$.

Let V_d denote the Del Pezzo surface of degree d (see [3] for the definition and some of its fundamental properties).

Theorem 2.4. *For the Del Pezzo surface V_d of degree d, $1 \leq d \leq 7$, regarded as embedded in a projective space by means of the linear system associated with the anticanonical divisor $-K_{V_d}$, the zeta function for rational curves $Z_{rat}(V_d\,;x)$ is given by*

$$Z_{rat}(V_7;x) = \frac{1+x-x^3+x^5+x^6-x^7}{1-x},$$

$$Z_{rat}(V_6;x) = \frac{1+2x-x^3+x^4+3x^5+2x^6-2x^7}{1-x},$$

$$Z_{rat}(V_5;x) = \frac{1+3x+x^3+5x^4+10x^5+5x^6-5x^7}{1-x},$$

$$Z_{rat}(V_4;x) = \frac{1+4x+11x^3+24x^4+40x^5+16x^6-16x^7}{1-x},$$

$$Z_{rat}(V_3;x) = \frac{1+5x+66x^3+144x^4+216x^5+72x^6-72x^7}{1-x},$$

$$Z_{rat}(V_2;x) = \frac{1+6x+569x^3+1440x^4+2016x^5+576x^6-576x^7}{1-x},$$

$$Z_{rat}(V_1;x) = \frac{1+8x+7x^2+3501x^3+12628x^4+21419x^5+15857x^6-3509x^8}{1-x^2}.$$

Remark. The Del Pezzo surfaces with $d = 8, 9$ have already been treated in Theorem 2.3.

3. Outline of the proof. The cases of the projective planes and the rational ruled surfaces cause no difficulty, since we know the structure of their Néron-Severi groups quite explicitly and we can assure the irreducibility of a divisor class by a simple inequality. In order to treat the Del Pezzo surfaces, however, we must pay special attention to the irreducibility of divisors. For the Del Pezzo surface V_4 of degree 4, for example, our task turns out to be equivalent to find all the integer solutions of the equation

$$8\sum_{1\leq i\leq 5} x_i^2 - 2\sum_{1\leq i<j\leq 5} x_i x_j - 2d\sum_{1\leq i\leq 5} x_i = d^2 - 9d + 18$$

for each positive integer d (= the degree of the rational curve under consideration) ≥ 3, subject to the conditions

$$x_1, x_2, x_3, x_4, x_5 \geq 0,$$
$$2x_1+2x_2-x_3-x_4-x_5 \leq d,\ 2x_1-x_2+2x_3-x_4-x_5 \leq d,$$
$$\cdots,\ -x_1-x_2-x_3+2x_4+2x_5 \leq d,$$
$$x_1+x_2+x_3+x_4+x_5 \leq 2d.$$

The equation corresponds to the condition that a divisor class $((\sum_{1\leq i\leq 5} x_i + d)/3)\ell_0 - \sum_{1\leq i\leq 5} x_i\ell_i$ on V_4 (in the notation of [3]) is of arithmetic genus zero, and the inequalities correspond to the condition that the divisor class contains an irreducible curve. The solutions to the problem above are found to be as in Table:

Table

d	$(x_1, x_2, x_3, x_4, x_5)$
3	(0, 0, 0, 0, 0), (1, 1, 1, 0, 0), (2, 1, 1, 1, 1)
4	(1, 1, 0, 0, 0), (2, 1, 1, 1, 0), (2, 2, 2, 1, 1)
5	(1, 0, 0, 0, 0), (2, 1, 1, 0, 0), (2, 2, 2, 1, 0), (3, 1, 1, 1, 1), (3, 2, 2, 2, 1)
6	(0, 0, 0, 0, 0), (2, 1, 0, 0, 0), (2, 2, 2, 0, 0), (3, 1, 1, 1, 0), (3, 2, 2, 2, 0), (3, 3, 3, 2, 1), (4, 2, 2, 2, 2)
odd ≥ 7	$(\frac{d+1}{2}, \frac{d-1}{2}, \frac{d-1}{2}, \frac{d-1}{2}, 1)$, $(\frac{d+1}{2}, 1, 1, 1, 1)$, $(\frac{d-1}{2}, \frac{d-1}{2}, \frac{d-1}{2}, \frac{d-3}{2}, 0)$, $(\frac{d-1}{2}, 1, 1, 0, 0)$, $(\frac{d-3}{2}, 0, 0, 0, 0)$
even ≥ 8	$(\frac{d}{2}, \frac{d}{2}, \frac{d}{2}, \frac{d}{2}-1, 1)$, $(\frac{d}{2}, 1, 1, 1, 0, 0)$, $(\frac{d}{2}, \frac{d}{2}-1, \frac{d}{2}-1, \frac{d}{2}-1, 0)$, $(\frac{d}{2}-1, 1, 1, 0, 0, 0)$

A similar type of Diophantine equations arise for other Del Pezzo surfaces. The lower the degree of the surface, the more complicated the equation to be solved. At first sight, these equations seem to be intractable, but fortunately enough (or for some deep reason which the author does not know), the integer solutions for them reside very near the boundary of the region defined by the inequalities. Hence all we have to do is to push the boundary walls inward step by step, and to pick up the solutions among the lattice points which hit the walls.

References

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