Uniqueness of Unibranched Curve in $\boldsymbol{R}^2$ up to Simple Blowings up

By Masanori KOBAYASHI*) and Tzee-Char KUO**)

(Communicated by Heisuke HIRONAKA, M. J. A., June 12, 1997)

Introduction. We consider the case of two real variables. A function $g:\boldsymbol{R}^2 \to \boldsymbol{R}$ is blow-analytic if there exists a composition of simple blowings up, each centered at a point, $\beta = \beta_1 \circ \cdots \circ \beta_n : X \to \boldsymbol{R}^2$ such that $g \circ \beta$ is analytic ([7], [4]). If a homeomorphism $h:\boldsymbol{R}^2 \to \boldsymbol{R}^2$ and its inverse, h^{-1}, both have blow-analytic components, we say h is blow-analytic.

S. Koike ([5]) was the first to discover a blow-analytic homeomorphism of $\boldsymbol{R}^3$ with itself which is not Lipschitz (see also [4]). L. Paunescu, in [9], has discovered one, also of $\boldsymbol{R}^3$, which does not preserve the multiplicity of analytic arcs.

On the other hand, M. Suzuki ([11]) and T. Fukui ([3]) have found some blow-analytic invariants which can be used to show, for example, that functions like x, $x^2 - y^3$, $x^3 - y^7$ are not blow-analytically equivalent. A seemingly simple question, raised by Koike, is whether a blow-analytic homeomorphism $h:\boldsymbol{R}^2 \to \boldsymbol{R}^2$ can carry a line $\{x = 0\}$ to a singular curve such as $\{x^2 = y^3\}$ or $\{x^3 = y^7\}$. (These are as topological spaces; analytic structures are ignored.)

We shall answer the question in the affirmative.

Let C be the germ of a singular curve in $\boldsymbol{R}^2$, which is unibranched in the sense that its complexification has only one branch ([12]).

We first take a sequence of blowings up to desingularize C to a smooth curve, $\tilde{C}$, transversal to exactly one exceptional curve, we then prove that there is a way to blow down to $\boldsymbol{R}^2$ again so that the images of $\tilde{C}$, at all stages, remain smooth. This is the content of the Main Theorem (Theorem 8).

*) Department of Mathematics, Tokyo Institute of Technology.

**) School of Mathematics and Statistics, University of Sydney.

The first author is partially supported by Fûjukai Foundation. The second author is partially supported by ARC Institutional Grant.

Our results and techniques are developped merely for the real analytic case. We no longer have them in the complex case, where things are too rigid.

Notation and an invariant for real exceptional curves. First we define an invariant for simple blowings up on surfaces.

Proposition 1. *There are only two real analytic equivalence classes of germs of compact connected smooth real analytic manifold of dimension one embedded in a smooth surface; that is,*

(1) *the direct product of the germ of a one dimensional open segment and* $\boldsymbol{RP}^1$ *or*

(2) *the tubular neighbourhood of the exceptional divisor of the blowing up of the origin of* $\boldsymbol{R}^2$.

Topologically (1) *is an annulus,* (2) *is a Möbius band.*

We say a neighbourhood of $\boldsymbol{RP}^1$ is *even* for (1) and *odd* for (2). We also call the central curve *even* and *odd* accordingly.

Definition 2. We call a blowing-up of a maximal ideal of a smooth point in a surface a *simple blowing up*.

The exceptional curve is an odd curve. A simple blowing up centered at a point on a curve changes the parity of the curve.

Proposition 3. *Any real analytic curve singularity embedded in a smooth surface can be resolved by a sequence of simple blowings up.*

Take a sequence of simple blowings-up $\beta = \beta_n \circ \cdots \circ \beta_1 : X = X_n \to X_0 = \boldsymbol{R}^2$, $\beta_i : X_i \to X_{i-1}$, whose centers are all above the origin of $\boldsymbol{R}^2$. The whole sequence is just the real slice of a natural composition of point blowings-up of $\boldsymbol{C}^2$. The exceptional set in X is a connected union of normal crossing real projective lines E_i which is the strict transform of the exceptional divisor of β_i. We associate to $(X, \{E_i\})$ a matrix whose entries are in $\boldsymbol{Z}/2\boldsymbol{Z}$.

Definition 4. The *intersection matrix* of $(X, \{E_i\})$ is an $n \times n$ matrix (a_{ij}) where

(1) a_{ii} is zero or one according to the parity of E_i and

(2) a_{ij} $(i \neq j)$ is one if E_i and E_j intersect or zero otherwise.

This is nothing but the modulo two object of the usual intersection matrix $((E_i. E_j))$ for a complexification. We can speak of the determinant of the intersection matrix, which is defined to be one for $\boldsymbol{R}^2$.

Proposition 5. *The determinant of the intersection matrix of the exceptional curves is invariant under simple blowings-up and down.*

Thus, the surface made by simple blowings up from $\boldsymbol{R}^2$ has a tree of $\boldsymbol{RP}^1$'s whose intersection matrix has determinant one.

By performing additional simple blowings up if necessary, one can assume that all components of the exceptional set are odd curves for the sake of simplicity.

We use a standard language of dual graphs; an exceptional real projective line corresponds to a vertex, and if two lines intersect at a point, the corresponding vertices are joined by an edge. In the following, a graph means a weighted graph, where each vertex is assigned a parity as its weight. We call a vertex corresponding to an odd curve a *cell*.

Remark. By translating the curve configuration into the language of graphs, one loses the information of the cyclic order of the double points on each real projective line. However, this does not affect the proof of the statement.

Note also that ratio of points on $\boldsymbol{RP}^1$ is not an invariant any more by Cartan's Theorem B ([2]; see also [6]).

We introduce the following four procedures, which are all composition of simple blowings up.

Definition 6. A *space station* is a graph arising from an iterated applications of the following four constructions to the null graph:

(1) *launch*: add a single cell without edge;

(2) *twin extension*: growing two cells from a mother cell;

(3) *rifle extension*: growing three cells in a row;

(4) *tick extension*: growing four cells filed in a row, with the second one being connected to a mother cell.

We use the word 'contraction' for the inverse of 'extension'.

A construction begins with (1). The graph is connected if and only if (1) is never used again. We are merely interested in connected space stations, which is a tree.

In a space station, there is at most one edge between two cells; each cell has weight 1.

We define the *determinant* of the graph to be that of the obvious intersection matrix. The determinant of the null graph is defined to be one. Note that the determinant does not depend on the numbering of cells.

Results. Proposition 7. (1) *Let S be a connected graph consisting of cells. Then S is a space station if and only if S is a tree and the determinant is one.*

(2) *Let S be a space station and c be any cell of S. Then by contractions* (2), (3) *and* (4), *S can become one of the following three space stations*: (A) c, (B) $c - o - o$ *or* (C) $o - c - o - o$, *where o's are the other cells.*

Regarding c as the exceptional component intersecting the strict transform of the original curve, we have the following

Theorem 8. *Let X be a two-dimensional smooth tubular neighbourhood of a tree of odd curves. Assume that the determinant of the intersection matrix is one. Then X can be obtained by a composition of simple blowings up $f: X \to \boldsymbol{R}^2$, whose exceptional set is the given tree.*

Moreover, choose any noncompact smooth curve C which cuts transversally only one of the odd curves. Then f can be chosen so that f_C is a real analytic isomorphism on every step of simple blowing down.

Theorem 9. *A germ of real unibranched curve in $\boldsymbol{R}^2$ is unique up to blow-analytic homeomorphism.*

Example 10. Here we give an explicit answer to Koike's question in the introduction.

(1) Case $\{x^2 = y^3\}$. By three simple blowings up, we have a good resolution of the curve, in the sense that the strict transform of the cuspidal curve is smooth, intersects only one exceptional curve and the intersection is transversal. One of the exceptional curve, which comes from the second blowing up, is even; thus we perform another simple blowing up at a general point of that curve. The resulting dual graph is nothing but the case (C) in

Proposition 7(2), thus we can blow down the strict transform to a line in $\boldsymbol{R}^2$.

(2) Case $\{x^3 = y^7\}$. By five simple blowings up, we reach a good resolution. The dual graph of the exceptional curves has linear five vertices and only the mid vertex is odd, which the strict transform of the original curve intersects. By two simple blowings up, each at the intersection of an end component and the next one, all the vertices become odd. Performing two rifle contractions, we arrive at the case (A).

Thus in each case, the singular curve is mapped to a line in $\boldsymbol{R}^2$ by a blow-analytic homeomorphism.

We have the following Paunescu-type result for $\boldsymbol{R}^2$.

Corollary 11. *A blow-analytic homeomorphism of $\boldsymbol{R}^2$ may not preserve the multiplicity of analytic arcs.*

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