Twist and Generalized Chebyshev Polynomials

By Fumio HAZAMA

Department of Natural Sciences, College of Science and Engineering, Tokyo Denki University

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1. Introduction. In the article [2], we investigated a possible generalization of the Chebyshev polynomials $T_n(x)$, $U_n(x)$ ($n = 0, 1, \cdots$), focusing on the Diophantine equation satisfied by them: (*) $T_n(x)^2 - (x^2 - 1)U_{n-1}(x)^2 = 1$, $n = 1, 2, \cdots$. The crucial idea of [loc. cit.] was to regard this as a defining equation of a twist of a conic by itself. As a natural generalization, we considered a twist of a conic by an arbitrary hyperelliptic curve, and obtained a family of Diophantine equations which have solutions in a certain one-parameter family of polynomials in one variable. In the present article, we proceed to higher dimensional cases and consider the twist of a double cover of the affine space of dimension $N \geq 1$ by itself. As a result, we find certain families of polynomials in $N$ variables, called the generalized Chebyshev polynomials, which enjoy a lot of fundamental properties similar to the ones the usual Chebyshev polynomials do. The purpose of this article is to announce these properties. Details will appear elsewhere.

2. Twist and generalized Chebyshev polynomials. Let $k$ be an arbitrary field of characteristic $\neq 2$. Let $G_m$ denote the multiplicative group and let $T_N$ denote the norm torus of dimension $N$. It is defined to be the kernel of the norm map $G_{m+1} \rightarrow G_m$ given by the formula: $(u_1, \cdots, u_{N+1}) \mapsto \Pi_{1 \leq i \leq N+1} u_i$. The norm torus $T_N$ is stable under the natural action of the symmetric group $S_{N+1}$ of degree $N + 1$ on $G_{m+1}$. Hence, if we denote by $A_{N+1}$ the alternating group of degree $N + 1$, then we have quotient maps: $T_N \rightarrow T_N/A_{N+1}$, $T_N \rightarrow S_{N+1}$. We denote by $\Delta(u_1, \cdots, u_{N+1})$ the difference product $\Pi_{1 \leq i < j \leq N+1} (u_i - u_j)$, and by $D(x_1, \cdots, x_N)$ its square: $D = D(x_1, \cdots, x_N) = (\Delta(u_1, \cdots, u_{N+1}))^2$. Then the quotient $T_N/A_{N+1}$ is defined by the equation $y^2 = D(x_1, \cdots, x_N)$, where $x_k (1 \leq k \leq N)$ denote the $k$-th elementary symmetric polynomial. The rational maps $p, q$ are given by the formulas:

$$p(u_1, \cdots, u_{N+1}) = (x_1, \cdots, x_N, \Delta),$$

$$q(x_1, \cdots, x_N, y) = (x_1, \cdots, x_N).$$

The $n$-th power endomorphism of $G_{m+1}$ induces the endomorphism $[n]$ of $T_N$, and it commutes with the action of $S_{N+1}$. Therefore we have the following commutative diagram:

$$\begin{array}{ccc}
T_N & \xrightarrow{[n]} & T_N \\
\downarrow p & & \downarrow p \\
T_N/A_{N+1} & \xrightarrow{[n]} & T_N/A_{N+1} \\
\downarrow \phi & & \downarrow \phi \\
T_N/S_{N+1} & \xrightarrow{[n]} & T_N/S_{N+1}.
\end{array}$$

(Here we used the same symbol $[n]$ for the induced maps). Let $T_N'$ denote the twist of $T_N/A_{N+1}$ by the quadratic extension $k(T_N/A_{N+1})/k(T_N/S_{N+1})$, where $k(X)$ denotes the rational function field of a variety $X$ defined over $k$. The twist $T_N'$ is defined over $k(T_N/S_{N+1}) \equiv k(x_1, \cdots, x_N)$ and its defining equation is given by the following:

$$T_N': D(x_1, \cdots, x_N)Y^2 = D(x_1, \cdots, x_N),$$

where the capital letters $X_1, \cdots, X_N, Y$ are regarded as variables (see [2] for the fundamental properties of twists). As for the set $T_N'(k(x_1, \cdots, x_N))$ of $k(x_1, \cdots, x_N)$-rational points of $T_N'$, we have the following theorem which can be proved in the same way as in [2]:

**Theorem 2.1.** There is a natural bijection between the set $T_N'(k(x_1, \cdots, x_N))$ and the set $\mathcal{A}(f \setminus \text{Rat}_k(T_N/A_{N+1}, T_N/A_{N+1}) \setminus f)$, where $\text{Rat}_k(V, W)$ for $k$-varieties $V, W$ denotes the set of $k$-rational maps of $V$ to $W$, and $\setminus f$ denotes the involution of $T_N/A_{N+1}$ defined by the formula $\iota(x_1, \cdots, x_N, y) = (x_1, \cdots, x_N, -y)$.

By this theorem, the $n$-th power map $[n]$ corresponds to a $k(x_1, \cdots, x_N)$-rational point on the twist $T_N'$, which we denote by

$$(t_1(\cdots)(x_1, \cdots, x_N), \cdots, t_N(\cdots)(x_1, \cdots, x_N), s_n(x_1, \cdots, x_N)).$$

We call $t_0(\cdots)(x_1, \cdots, x_N)$ the generalized Chebyshev polynomial of the first kind, and $s_n(x_1, \cdots, x_N)$ the generalized Chebyshev polynomial of the second kind, because of the following natural generalization of (*):
Theorem 2.2. For any positive integer \( n \), the point
\[
(X_1, \ldots, X_N, Y) = (t^{(1)}_n(x_1, \ldots, x_N), \ldots, t^{(N)}_n(x_1, \ldots, x_N), s_n(x_1, \ldots, x_N))
\]
lies on the twist
\[
T'_n : D(x_1, \ldots, x_N) Y^2 = D(X_1, \ldots, X_N).
\]

Note that the generalized Chebyshev polynomials of the first kind are expressed as \( t^{(i)}_n = m_{(\lambda)} \), where \( m_{(\lambda)} = \sum u_1^{(i)} \cdots u_N^{(i)} \) with \( (\lambda_1, \ldots, \lambda_N) \) a Young diagram, and the sum going through all the distinct permutations of \( (\lambda_1, \ldots, \lambda_N) \). In view of this, we call \( m_{(\lambda)} \) the generalized Chebyshev polynomials.

3. Various properties. The usual Chebyshev polynomials are known to have a lot of fundamental properties. Most of these can be generalized as follows.

**Theorem 3.1 (Multiplicativity).** For any positive integers \( m, n \),
\[
\begin{align*}
t^{(m)}_n((x_1, \ldots, x_N)) &= t^{(m)}_n(x_1, \ldots, x_N), \\
t^{(N)}_m((x_1, \ldots, x_N)) &= t^{(N)}_m(x_1, \ldots, x_N), \\
s_m((x_1, \ldots, x_N)) &= s_m(x_1, \ldots, x_N), \\
s_n((x_1, \ldots, x_N)) &= s_n(x_1, \ldots, x_N).
\end{align*}
\]

For any positive integer \( m \), let \( \zeta_m \) denote a primitive \( m \)-th root of unity, and let \( \mu_m \) denote the group of \( m \)-th roots of unity. Then the set
\[
Z \text{ of the common zeroes of } N \text{ polynomials } t^{(1)}_n(x_1, \ldots, x_N)
\]
found to be
\[
\{ (x_1, \ldots, x_N) ; x_1 = t^{(1)}_n(u_1, \ldots, u_{N+1}), \text{ with } u_i \in \mathbb{C} \text{ for } 1 \leq i \leq N, \text{ and } \prod_{1 \leq i \leq N+1} u_i = 1 \}.
\]

**Theorem 3.2 (Discrete orthogonality).** Let \( N \geq 2 \) and let \( k, l \) be positive integers. If \( N \) is even, then
\[
\sum_{(x_1, \ldots, x_N) \in Z} t^{(k)}_n(x_1, \ldots, x_N) t^{(l)}_n(x_1, \ldots, x_N) = \begin{cases} n^N(N + 1)^2, k \equiv l \equiv 0 \pmod{(N + 1)n}, \\
N^N(N + 1), k + l \equiv 0 \pmod{(N + 1)n}, \\
k \neq 0 \pmod{n}, \\
l \neq 0 \pmod{n}, \\
0, \text{ otherwise.}
\end{cases}
\]

If \( N \) is odd, then
\[
\sum_{(x_1, \ldots, x_N) \in Z} t^{(k)}_n(x_1, \ldots, x_N) t^{(l)}_n(x_1, \ldots, x_N) = \begin{cases} n^N(N + 1)^2, k \equiv l \equiv 0 \pmod{(N + 1)n), \\
N^N(N + 1), k + l \equiv 0 \pmod{(N + 1)n), \\
k \neq 0 \pmod{n}, & \text{and or} \\
l \neq 0 \pmod{n}, \\
0, \text{ otherwise.}
\end{cases}
\]

For a complex valued continuous function \( f \) on \( S \), let \( \| f \| = \max_{x \in S} |f(x)| \).

**Theorem 3.3 (Orthogonality).** Let \( S \) denote the image of \( \{ (u_1, \ldots, u_{N+1}) \in (\mathbb{C}^*)^{N+1} \} \) under the map \( (u_1, \ldots, u_{N+1}) \mapsto (x_1, \ldots, x_N) \). Then for any pair \( m_{(\lambda)}, m_{(\mu)} \) of the generalized Chebyshev polynomials,
\[
\int_S m_{(\lambda)}(x_1, \ldots, x_N) m_{(\mu)}(x_1, \ldots, x_N) \frac{dx_1 \cdots dx_N}{D(x_1, \ldots, x_N)} = \begin{cases} \frac{2\pi i}{N+1} d(\lambda), & \text{if } (\lambda_1, \lambda_2 + \mu_N, \ldots, \lambda_N) \\
 + \mu_2, \mu_1 = c \cdot 1, \ldots, 1, & \text{for some integer } c, \\
0, \text{ otherwise.}
\end{cases}
\]

For a complex valued continuous function \( f \) on \( S \), let \( \| f \| = \max_{x \in S} |f(x)| \).

**Theorem 3.4 (Extremal property).** For any monic polynomials \( p \) of degree \( n > 0 \) in \( N \) variables \( x_1, \ldots, x_N \), we have \( \| p \| \geq n + 1 \). The equality holds if and only if \( p = t^{(1)}_n \) or \( p = t^{(N)}_n \).

The following is a direct consequence of this theorem:

**Corollary ([1], Theorem).** For polynomials \( P_{n-1}(z, \overline{z}) \) of degree \( n - 1 \) over \( \mathbb{C} \),
\[
\inf_{n-1} \max_{z \in S} |z^n + P_{n-1}(z, \overline{z})| = 3.
\]
Moreover, the maximum is attained only when \( z^n + P_{n-1}(z, \overline{z}) = t^{(1)}_n(z, \overline{z}) \).

**Remark.** Our theory gives a much shorter and simpler proof of this corollary than the one in [loc. cit.]. Furthermore, if we put \( N = 1 \) in our proof, then we obtain much simpler proof of [3], Theorem 2.1 which deals with the extremal property of the usual Chebyshev polynomials.

**Theorem 3.5 (Differential equation).** For any Young diagram \( (\lambda) = (\lambda_1, \ldots, \lambda_N) \), the corresponding polynomial \( m_{(\lambda)} \) is a solution of the partial differential equation
\[
\sum_{1 \leq k \leq N} A_{kk} f_{x_k} x_k + \sum_{1 \leq k < l \leq N} A_{kl} f_{x_k} x_l + \sum_{1 \leq k < l \leq N} B_{kl} f_{x_k} x_l = (N \sum_{1 \leq i \leq N} \lambda_i^2 - 2 \sum_{1 \leq i < j \leq N} \lambda_i \lambda_j) f
\]
where
\[
A_{kk} = (-k^2 + (N + 1)k)x_k^2 - (N + 1)
\]
\[
\sum_{0 \leq r \leq k-1} (2k - 2r)x_{2k-r}, \quad 1 \leq k \leq N,
\]
\[
A_{kl} = (2k((N + 1) - l))x_kx_l - 2(N + 1)
\]
\[
\sum_{0 \leq r \leq k-1} (k + l - 2r)x_{k+l-r}, \quad 1 \leq k < l \leq N,
\]
\[
B_k = k(N + 1 - k)x_k, \quad 1 \leq k \leq N.
\]
Let \( Q_N(\lambda) = Q_N(\lambda_1, \ldots, \lambda_N) \) denote the quadratic form \( N \sum_{1 \leq i < j \leq N} \lambda_i \lambda_j \).

**Theorem 3.6 (Uniqueness).** For any \( E \in \mathbb{Z} \), the set \( \{m Q_N(\lambda): E\} \) of the generalized Chebyshev polynomials constitutes a base of the vector space of polynomial solutions of the partial differential equation \( Df = Ef \), where \( D \) denotes the differential operator appearing on the left hand side of (3.5.1).

**Theorem 3.7 (Dimension formula).** The dimension of the space of polynomial solutions of (3.5.1) is
\[
\begin{cases}
0, & \text{if } E \text{ is odd}, \\
\sum_{d \mid E/2} \left( \frac{d}{3} \right), & \text{if } E \text{ is even and } E/2 \text{ is not a square}, \\
\sum_{d \mid E/2} \left( \frac{d}{3} \right) + 1, & \text{if } E \text{ is even and } E/2 \text{ is a square},
\end{cases}
\]
where \(- \) denotes the Legendre symbol.

**Theorem 3.8 (Numerical integration I).** For each pair \((i, j)\) of integers with \( 0 \leq i, j \leq n - 1 \), let \( P(i, j) = (\zeta^{3i}, \zeta^{3j+1}, \zeta^{3j-3i+1}) \). Then for any polynomial \( f \) in two variables \( x_1, x_2 \) of degree \( \leq 2n - 1 \), the following numerical integration formula holds:
\[
\int_s f \frac{dx_1dx_2}{\sqrt{D(x_1, x_2)}} = - \frac{2\pi^2}{3n^2} \sum_{0 \leq i, j \leq n-1} f(P(i, j)).
\]

**Theorem 3.9 (Numerical integration II).** For each triple \((i, j, k)\) of integers with \( 0 \leq i, j, k \leq n - 1 \), let \( P(i, j, k) = (\zeta^{3i}, \zeta^{3j+1}, \zeta^{3k+5}, \zeta^{3k-3i+1}, \zeta^{3k-3j+1}) \). Then for any polynomial \( f \) in three variables \( x_1, x_2, x_3 \) of degree \( \leq 2n - 1 \), the following numerical integration formula holds:
\[
\int_s f \frac{dx_1dx_2dx_3}{\sqrt{D(x_1, x_2, x_3)}} = - \frac{\pi^3}{3n^2} \sqrt{-1} \sum_{0 \leq i, j, k \leq n-1} f(P(i, j, k)).
\]

Let \( t_n \) denote the self map of \( S \) defined by the formula: \( t_n(x) = (t_n^{(1)}(x), \ldots, t_n^{(n)}(x)) \) for \( x = (x_1, \ldots, x_n) \in S \). Let \( B \) denote the family of Borel subsets of \( S \), and let \( \mu \) be the measure defined by
\[
\mu(B) = \frac{N + 1}{(2\pi^2 - 1)^n} \int_B \frac{dx_1 \cdots dx_n}{\sqrt{D(x_1, \ldots, x_n)}}, \quad B \in B.
\]
Then we have the following:

**Theorem 3.10 (Ergodicity).** Each \( t_n \) \( (n \geq 1) \) preserves the measure \( \mu \). Moreover, each \( t_n \) \( (n > 1) \) is ergodic.

**References**

