The Dynamics of Nearly Abelian Polynomial Semigroups at Infinity

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Abstract: We prove that a nearly abelian polynomial semigroup has the simultaneously normalizing coordinate in the neighborhood of infinity. This result has been expected by A. Hinkkanen and G. J. Martin as Conjecture 7.1 in [5].

We begin this paper with some definitions, which is given by [5].

Definition. A *polynomial semigroup* G is a semigroup generated by a family of non-constant polynomial functions on $\boldsymbol{C} \cup \{\infty\}$ to itself. Here the semigroup operation is functional composition. And let $\langle f_1, \ldots, f_n, \ldots \rangle$ denote the semigroup generated by $f_1, \ldots, f_n, \ldots$. If G is a polynomial semigroup, we define the set of normality $N(G)$ as following;

$N(G) = \{z \in \boldsymbol{C} \cup \{\infty\}$: there is a neighborhood V of z such that $G|_V$ is a normal family with respect to the spherical metric$\}$.

Definition. A polynomial semigroup is *nearly abelian* if there is a compact family Φ of Möbius transformations with the following properties;

(i) $\phi(N(G)) = N(G)$ for all $\phi \in \Phi$, and

(ii) for all $f, g \in G$, there is a $\phi \in \Phi$ such that $f \circ g = \phi \circ g \circ f$.

Next we state our main theorem.

Theorem 1. *If G is a nearly abelian polynomial semigroup and G contains some polynomials of degree at least two, then there is a neighborhood of ∞ on which G is analytically conjugate to a subsemigroup of $\langle z \mapsto az^n : |a| = 1, n = 1,2,3, \ldots \rangle$.*

Remark. The condition that "G contains some polynomials of degree at least two" cannot be removed. In fact, there are counterexamples to the assertion without it. A simple example is $\langle z \mapsto 2z \rangle$.

We need two lemmas to prove Theorem 1. The first one is a consequence of Theorem 4.1 in [5].

Lemma 2. *Let G be a nearly abelian polynomial semigroup. Then for each $g \in G$ of degree at least two, we have $N(G) = N(\langle g \rangle)$.*

The next lemma is connected with the Böttcher function. (see [3] for the proof).

Lemma 3. *Suppose that f is a polynomial of degree n which is at least two. Then there exist a neighborhood V of ∞ and an injective holomorphic map $\varphi : V \mapsto \boldsymbol{C} \cup \{\infty\}$ such that*

(*i*) $\varphi(\infty) = \infty$,

(*ii*) $\displaystyle\lim_{z\to\infty} \frac{\varphi(x)}{z} = 1$,

(*iii*) $\varphi \circ f \circ \varphi^{-1}(\zeta) = a\zeta^n$, *where* $\zeta \in \varphi(V)$ *and* $a = \displaystyle\lim_{z\to\infty} \frac{f(z)}{z^n}$, *and*

(*iv*) *if Ω is the connected component of $N(\langle f \rangle)$ including ∞, then the map $z \mapsto \log|\varphi(z)|$ coincides with the Green function of Ω having the pole at ∞.*

Proof of Theorem 1. Let g be an element of G with degree n which is at least two. Then there is a Möbius transformation τ with the property that

$$\lim_{z\to\infty} \frac{\tau \circ g \circ \tau^{-1}(z)}{z^n} = 1.$$

It is sufficient to prove this theorem that we prove the similar assertion to $\tau \circ G \circ \tau^{-1}$. Therefore we may suppose that there is a $g \in G$ such that

$$\lim_{z\to\infty} \frac{g(z)}{z^n} = 1.$$

Using Lemma 3, we obtain a neighborhood V of ∞ and injective holomorphic map $\varphi_g : V \rightarrow \boldsymbol{C} \cup \{\infty\}$ such that

$$\varphi_g(\infty) = \infty,$$
$$\lim_{z\to\infty} \frac{\varphi_g(z)}{z} = 1,$$

and

$$\varphi_g \circ g \circ \varphi_g^{-1}(\zeta) = \zeta^n,$$

where $\zeta \in \varphi_g(V)$.

Suppose that f is another element of G of degree m which is at least two. Again using Lemma 3 and taking a smaller V if necessary, we can find a map $\varphi_f : V \to \boldsymbol{C} \cup \{\infty\}$ such that

$$\varphi_f(\infty) = \infty,$$

$$\lim_{z\to\infty} \frac{\varphi_f(z)}{z} = 1,$$

and

$$\varphi_f \circ f \circ \varphi_f^{-1}(\zeta) = a\zeta^m,$$

where

$$a = \lim_{z\to\infty} \frac{f(z)}{z^m}.$$

It follows from Lemma 2 that $N(\langle f \rangle) = N(\langle g \rangle)$ and hence the components of $N(\langle f \rangle)$ and $N(\langle g \rangle)$ containing ∞ also coincide. From the uniqueness of the Green function.

$$\log|\varphi_g(z)| = \log|\varphi_f(z)|.$$

Here φ_g/φ_f is a meromorphic function and $|\varphi_g/\varphi_f| = 1$. So the maximum principle says that φ_g/φ_f is a constant function. The condition

$$\lim_{z\to\infty} \frac{\varphi_g(z)}{z} = \lim_{z\to\infty} \frac{\varphi_f(z)}{z} = 1$$

gives $\varphi_f = \varphi_g$.

Let Φ be the set of the Möbius transformations given in the definition of a nearly abelian semigroup, then there exists a $\sigma \in \Phi$ such that $f \circ g = \sigma \circ g \circ f$ and $\sigma(N(G)) = N(G)$. Because $N(G)$ contains ∞,

$$\left| \lim_{z\to\infty} \frac{\sigma(z)}{z} \right| = 1.$$

And from $f \circ g = \sigma \circ g \circ f$, we get $|a| = |a^n|$, so $|a| = 1$.

After all, any $f \in G$ of degree at least two is conjugate to an element of $\langle z \mapsto az^n : |a| = 1, n = 1,2,3,\ldots \rangle$ near ∞ by a function $\varphi = \varphi_f = \varphi_g$.

Finally, we shall consider the remaining case when $f \in G$ is a polynomial of degree one. From the same reason as is in the preceding case,

$$\left| \lim_{z\to\infty} \frac{f(z)}{z} \right| = 1.$$

Generally, we have $f(N(G)) \subseteq N(G)$. So if we denote by Ω the component of $N(G)$ containing ∞, then $f(\Omega) \subseteq \Omega$. From the Schwarz lemma, we have gotten $f(\Omega) = \Omega$. And it implies

$$\log|\varphi(f(z))| = \log|\varphi(z)|,$$

which is the invariance of the Green function. From this equation we can conclude that $\varphi \circ f \circ \varphi^{-1}(z) = az$ where $a = \lim_{z\to\infty} \frac{f(z)}{z}$. $\square$

References

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