Complete Submanifolds with Parallel Mean Curvature in a Sphere

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Abstract: In this paper, we prove an intrinsic rigidity theorem for complete submanifolds with parallel mean curvature in a sphere, which generalize the results due to Alencar, do Carmo and Santos.

Key words: Parallel mean curvature; sphere; complete submanifold.

1. Introduction. Let M^n be an n-dimensional oriented manifold immersed in an $(n+p)$-dimensional unit sphere S^{n+p}, with mean curvature H and second fundamental form B. We put $\phi(X, Y) = B(X, Y) - \langle X, Y \rangle H$ for any tangent vector fields X and Y on M^n. Assume the mean curvature of M^n is parallel, we denote by B_H the square of the positive root of

$$x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|x - n(|H|^2 + 1).$$

Alendar and do Carmo [1] proved that if M^n is compact, $p = 1$ and $|\phi|^2 \le B_H$, then either $|\phi|^2 = 0$ and M^n is totally umbilic or $|\phi|^2 = B_H$ and M^n is a Clifford torus or an $H(r)$-torus of appropriate radii. Later, Santos [3] generalized this result to submanifolds.

The purpose of this paper is to generalize the results of Alendar, do carmo and Santos to complete submanifolds. Following Santos [3] we denote by ϕ_H the bilinear map defined by

$$\phi_h(X, Y) = \langle \phi(X, Y), H \rangle,$$

and by $B_{p,H}$ the function of p and H given by

$$B_{p,H} = \begin{cases} 1/(2 - 1/p), & \text{if } p = 1 \text{ or } h = 0, \\ 1/(2 - (1/(p-1))), & \textit{otherwise.} \end{cases}$$

We denote $C_{p,H}$ by

$$(1) \quad C_{p,H} = B_{p,H}\{n(1 + |H|^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}|\phi_H|\}.$$

The main result of the present paper is the following:

Theorem. Let M^n be a complete submanifold with parallel mean curvature $H(\neq 0)$ in S^{n+p}. Then either M^n is a totally umbilical sphere, or $\mathit{sup}\,|\phi|^2 \ge C_{p,H}$.

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2. Preliminaries. Let M^n be a submanifold of S^{n+p}. Choose a local orthonormal frame field $\{e_1, \ldots, e_{n+p}\}$ in S^{n+p} such that restricting to M^n, $\{e_1, \ldots, e_n\}$ are tangent to M^n. The mean curvature H of M^n is defined by

$$H = \frac{1}{n}\sum_{i=1}^{n} B(e_i, e_i),$$

and the square of the second fundamental form B is defined by

$$S = \sum_{i,j=1}^{n} |B(e_i, e_j)|^2.$$

It is easy to see that the square length of the tensor ϕ is given by

$$|\phi|^2 = S - n|H|^2.$$

Now we assume that M^n is a subamnifold with parallel mean curvature H. From [1] and [3], we have

$$(2) \quad \frac{1}{2}\Delta|\phi|^2 \ge |\phi|^2\{n(1 + H^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}|\phi_H| - \frac{1}{B_{p,H}}|\phi|^2\},$$

where h^α_{ij} are the components of the second fundamental form and h^α_{ijk} are the covariant derivatives of h^α_{ij}.

The following two lemmas are important in this paper.

Lemma 1 [2]. Let M^n be a subamnifold in S^{n+p}, and let Ric denote the minimum Ricci curvature at each point. Then

$$Ric \ge \frac{n-1}{n}(-|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + n(|H|^2 + 1)).$$

Proof. It follows immediately from the main theorem of [2] and the equation $|\phi|^2 = S - n|H|^2$.

Lemma 2 [4]. Let M^n be a complete

Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function bounded from above on M^n. Then there exists a sequence $\{p_m\}$ of points in M^n such that

$$lim\, f(p_m) = sup\, f,\ lim|grad\, f|(p_m) = 0,$$
$$lim\ sup\ \Delta f(p_m) \leq 0.$$

3. Proof of theorem. Assume that M^n is not totally umbilical and $sup|\phi|^2 < C_{p,H}$. Then, from Lemma 1 we know that the Ricci curvature of M^n is bounded from below. Since ϕ^2 is bounded from above. Hence, from Lemma 2 there exists a sequence $\{p_m\}$ in M^n such that

$$lim|\phi|^2(p_m) = sup|\phi|^2 \tag{3}$$

and

$$lim\ sup\ \Delta|\phi|^2(p_m) \leq 0. \tag{4}$$

Since $|\phi|^2$ is bounded, $\{|\phi|^2(p_m)\}$ is a bounded sequence. Therefore, there exists a subsequence $\{p_{m'}\}$ of $\{p_m\}$ such that

$$lim\,|\phi|^2(p_{m'}) = t^2 \tag{5}$$

for some $t \geq 0$. Taking into account (3), (4), and (5), the inequality (2) gives rise to the following inequality

$$sup\,|\phi|^2(n(1+H^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}|\phi_H| - \frac{1}{B_{p,H}}t^2) \leq 0. \tag{6}$$

Since M^n is not totally umbilical, we get $t^2 \geq B_{p,H}\ \{n(1+|H|^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}\ |\phi_H|\} = C_{p,H}$, which contradicts our assumption. This completes the proof.

From the theorem we have

Corollary 1. Let M^n be a complete submanifold with parallel mean curvature $H\,(\neq 0)$ in S^{n+p}. If $|\phi|^2 = constant$, then either M^n is a totally umbilical sphere, or $|\phi|^2 \geq C_{p,H}$.

References

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