# A Remark on Naive Height of a Polarized Abelian Variety and its Applications

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**Introduction.** The aim of the present paper is to give an inequality between certain heights of isogeneous polarized abelian varieties defined over a number field (Theorem 0.1 below). As an application we obtain a generalization (Theorem 0.3 below) of the theorem of Masser and David concerning the number of rational points of small height on a simple polarized abelian variety (Theorem 0.5 below).

Let $A$ be a $g$-dimensional abelian variety defined over a number field $k$. Let $\mathcal{M}$ be a very ample line bundle of degree $d$ over $A$. Then $(A, \mathcal{M})$ determines a polarized abelian variety. By extending the base field if necessary, we have a *theta-structure* on $(A, \mathcal{M})$ (see [5, p. 297]). When a theta-structure $s$ is fixed, a basis $(\theta_{s,i})_{i=1}^{d}$ for the $k$-vector space $\Gamma(A, \mathcal{M})$ of global sections of $\mathcal{M}$ is uniquely determined up to a constant (see Section 1 below), hence determines an embedding of $A$ into the $(d-1)$-dimensional projective space $\boldsymbol{P}^{d-1}$. The *naive height* $h_{\mathrm{n}}$ of the triple $(A, \mathcal{M}, s)$ is defined by the absolute logarithmic Weil height of the $k$-valued point $(\theta_{s,i}(0))_{i=1}^{d}$ in $\boldsymbol{P}^{d-1}$.

Throughout this paper, $k$ denotes a number field of finite degree $\Delta = [k : \boldsymbol{Q}]$.

The fundamental result of this paper is the following. (The superscript below indicates the inverse image of a line bundle by a morphism [8, p. 110].)

**Theorem 0.1.** *Let $A$ and $B$ be $g$-dimensional abelian varieties over $k$, $f$ be an isogeny of $A$ onto $B$, and $\mathcal{M}$ and $\mathcal{N}$ be very ample line bundles over $A$ and $B$, respectively, such that $\mathcal{M} \simeq f^*\mathcal{N}$. For a theta-structure $t$ on $(B, \mathcal{N})$ which* is compatible *with a theta-structure $s$ on $(A, \mathcal{M})$, we have*

$$h_{\mathrm{n}}(A, \mathcal{M}, s) \geq h_{\mathrm{n}}(B, \mathcal{N}, t).$$

The exact meaning of the compatibility of theta-structures is defined in Section 1.

**Remark 0.2.** For the Faltings stable height $h_{\mathrm{st}}$, we know

$$h_{\mathrm{st}}(A) \geq h_{\mathrm{st}}(B) - \frac{1}{2}\log \deg f \quad \text{cf. [2, Lemma 5].}$$

□

Two theorems below are main applications of Theorem 0.1. We denote by $q_A$ the quadratic part of a Néron-Tate height on $A$. We define the *naive height* $h_{\mathrm{n}}$ of the pair $(A, \mathcal{M})$ by the minimum of $h_{\mathrm{n}}(A, \mathcal{M}, s)$. We know that the number of theta-structures is finite (see [5, p. 297]). The rotation $^{\otimes 4}$ means the tensor product of 4 copies of a line bundle [8, p. 153].

**Theorem 0.3.** *Let $\mathcal{L}$ be an arbitrary ample line bundle over $A$ and set $\mathcal{M} := (\mathcal{L} \otimes (-1)^* \mathcal{L})^{\otimes 4}$. Assume that $A$ is* simple *and a theta-structure on $(A, \mathcal{M})$ is defined over $k$. There exists a positive constant $C = C(g)$ such that for any finite extension field $F$ of $k$ of degree $D = [F : k]$ we have*

$$\# \{P \in A(F) \mid q_A(\mathcal{L}, P) < \frac{1}{C \Delta D}\} < C \deg \mathcal{L} \cdot h_{\mathrm{n}}(A, \mathcal{M})^{3g/2} \Delta^{3g/2} (1 + \log \Delta)^g D^g (1 + \log D)^g.$$

**Theorem 0.4.** *Under the same assumptions as those of Theorem 0.3 we have a positive constant $C = C(g)$ such that*

$$\min_{A(F) \ni P:\text{non-torsion}} q_A(\mathcal{L}, P) > \frac{C}{h_{\mathrm{n}}(A, \mathcal{M})^{3g} \Delta^{3g+1} (1 + \log \Delta)^{2g} D^{2g+1} (1 + \log D)^{2g}}.$$

The proof of these theorems is based on the next theorem 0.5 due to Masser [3] and David [1], which is a special case of Theorem 0.3. It seems difficult to us to generalize directly the method of [1] to prove them, but as we shall show below, they follow easily from our theorem 0.1.

**Theorem 0.5** (Masser-David). *Let $B$ be a $g$-dimensional* simple *abelian variety over a number field $k$ and $\mathcal{N}$ be an ample line bundle over $B$ of de-*

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gree $8^g$ *of type* $(8,\cdots,8)$. *Suppose that a theta-structure on* $(B, \mathcal{N})$ *is defined* over the base field $k$. *Then there is a positive constant* $C = C(g)$ *such that*

$$\# \{Q \in B(F) \mid q_B(\mathcal{N}, Q) < \frac{1}{C \Delta D}\}$$
$$< C \cdot h_n(B, \mathcal{N})^{3g/2} \Delta^{3g/2} (1 + \log \Delta)^g D^g (1 + \log D)^g.$$

As for elliptic curves, see [4], too.

**1. Naive height of a polarized abelian variety.** Let $k$ be a number field and $A$ be a $g$-dimensional abelian variety over $k$. We denote by $\mathcal{M}$ a very ample line bundle over $A$ of type $\delta$ [5, p. 294], where $\delta$ is a finite sequence $(d_1, \ldots, d_g)$ of positive rational integers $d_i$ such that $d_{i+1}$ divides $d_i$. Let $\mathcal{G}(\mathcal{M})$ be the theta group associated with $\mathcal{M}$ [5, p. 289]. It is a group scheme over $k$ [7, p. 225] which acts on $\mathcal{M}$. We assume that *a primitive* $d_1$*-th root of unity is already in* $k$ *and the group* $\mathcal{G}(\mathcal{M})(k)$ *of* $k$*-rational points of* $\mathcal{G}(\mathcal{M})$ *is isomorphic to the group* $\mathcal{G}(\delta)(k)$ below, in which case we say a theta-structure *is defined over* $k$.

Let $K(\delta)(k) := \oplus_{i=1}^{g} d_i^{-1}\boldsymbol{Z}/\boldsymbol{Z}$ and $\hat{K}(\delta)(k) := \mathrm{Hom}\,(K(\delta)(k), \boldsymbol{G}_m(k))$. As set, the group $\mathcal{G}(\delta)(k)$ equals $\boldsymbol{G}_m(k) \times \hat{K}(\delta)(k) \times K(\delta)(k)$. It acts naturally on a finite dimensional $k$-vector space $V(\delta)(k) := \mathrm{Map}\,(K(\delta)(k), \boldsymbol{A}^1(k))$, which induces a multiplication law on $\mathcal{G}(\delta)(k)$ [5, pp. 294-297]. We see that the abelian groups $K(\delta)(k)$ and $\hat{K}(\delta)(k)$ are subgroups of $\mathcal{G}(\delta)(k)$. An isomorphism $s : \mathcal{G}(\mathcal{M})(k) \simeq \mathcal{G}(\delta)(k)$ is called a *theta-structure* [5, p. 297]. Via $s$, the group $\mathcal{G}(\delta)(k)$ acts also on the $k$-vector space $\Gamma(A, \mathcal{M})$ of global sections of $\mathcal{M}$.

**Proposition 1.1.** *Once a theta-structure* $s$ *is fixed, the* $k$*-vector space* $\Gamma(A, \mathcal{M})$ *is isomorphic to the* $k$*-vector space* $V(\delta)(k)$ *as* $\mathcal{G}(\delta)(k)$*-modules. The isomorphism is unique up to multiplication by a constant in* $k$.

*Proof.* [5, pp. 295-297]. □

Let $Q_x$ be an element of $V(\delta)(k)$ defined as

$$Q_x(y) := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

The set $(Q_x)_{x \in K(\delta)(k)}$ forms a basis of the $k$-vector space $V(\delta)(k)$. Let $(\theta_{s,x})_{x \in K(\delta)(k)}$ be a subset of $\Gamma(A, \mathcal{M})$ that consists of the elements which correspond to $Q_x$ under an isomorphism in Proposition 1.1.

**Definition 1.2.** The *naive height* $h_n$ *of* $(A, \mathcal{M}, s)$ is the absolute logarithmic Weil height of the $k$-valued point $(\theta_{s,x}(0))_{x \in K(\delta)(k)}$ in the projective space $\boldsymbol{P}^{d-1}$, where $d$ is the dimension of $\Gamma(A, \mathcal{M})$ which is equal to deg $\mathcal{M}$.

By Proposition 1.1, the real number $h_n(A, \mathcal{M}, s)$ is well-defined.

**Remark 1.3.** At Archimedean places the values $\theta_{s,x}(0)$ are the classical *Thetanullwerte* $\theta_{mn}(\tau, 0)$. □

Let $H$ be any subgroup of $\mathcal{G}(\mathcal{M})(k)$ such that $s(H) \hookrightarrow \hat{K}(\delta)(k) \subset \mathcal{G}(\delta)(k)$ for a fixed theta-structure $s$. We divide $A$ and $\mathcal{M}$ by $H$; indeed, there exist an isogeny $f$ of $A$ onto an abelian variety $B$ over $k$ and an ample line bundle $\mathcal{N}$ over $B$ such that $\deg f = \# H$, $f^*\mathcal{N} \simeq \mathcal{M}$, and the $k$-vector space $\Gamma(B, \mathcal{N})$ is identified with the $H$-invariant subspace of $\Gamma(A, \mathcal{M})$ under $f^*$ [5, pp. 290-291].

Let $\mathcal{G}(\mathcal{M})(k)^*$ be the normalizer of $H$ in $\mathcal{G}(\mathcal{M})(k)$ and $L := K(\delta)(k) \cap s(\mathcal{G}(\mathcal{M})(k)^*)$. The group $\mathcal{G}(\mathcal{M})(k)^*$ acts naturally on

$$\Gamma(B, \mathcal{N}) \simeq \Gamma(A, \mathcal{M})^H.$$

In fact, we have [5, p. 291]

$$\mathcal{G}(\mathcal{M})(k)^*/H \simeq \mathcal{G}(\mathcal{N})(k).$$

On the other hand, we see

$$s(\mathcal{G}(\mathcal{M})(k)^*/H) = \boldsymbol{G}_m(k) \times \hat{K}(\delta)(k)/s(H) \times L$$

and we have

$$\hat{K}(\delta)(k)/s(H) \simeq \mathrm{Hom}\,(L, \boldsymbol{G}_m(k))$$

by definition. This leads to the existence of a theta-structure $t$ on $(B, \mathcal{N})$.

**Definition 1.4.** The theta-structures $s$ on $(A, \mathcal{M})$ and $t$ on $(B, \mathcal{N})$ *are compatible* if $B$ and $\mathcal{N}$ are the quotients of $A$ and of $\mathcal{M}$, respectively, by a subgroup of $s^{-1}(\hat{K}(\delta)(k))$ and if $t$ is induced by $s$ taking the subquotients.

**Proof of Theorem 0.1.** Let $(\theta_{s,x})_{x \in K(\delta)(k)}$ and $(\theta_{t,y})_{y \in L}$ be sets of global sections of $\mathcal{M}$ and $\mathcal{N}$, respectively, as described before. Note that $L$ is a subgroup of $K(\delta)(k)$. By using Theorem 4 of [5, p. 302], there is a constant $\lambda \in k$ such that $(\lambda \cdot \theta_{t,y})_{y \in L}$ is a subset of $(\theta_{s,x})_{x \in K(\delta)(k)}$ via $f^* : \Gamma(B, \mathcal{N}) \hookrightarrow \Gamma(A, \mathcal{M})$. The definition of the Weil height yields the proof. □

**2. Proofs of Theorem 0.3 and Theorem 0.4.**

**Proof of Theorem 0.3.** Fix a theta-structure $s$ of $\mathcal{G}(\mathcal{M})(k)$ satisfying

$$h_n(A, \mathcal{M}, s) = h_n(A, \mathcal{M}).$$

As indicated in the previous section, we have an isogeny $f : A \to B$, an ample line bundle $\mathcal{N}$ over $B$ of type $(8, \cdots, 8)$, and a theta-structure $t$ of $\mathcal{G}(\mathcal{N})(k)$ which is compatible with $s$ such that

$$\deg f = \deg \mathscr{L} \quad \text{and} \quad f^*\mathscr{N} \simeq \mathscr{M}.$$

As a line bundle of type $(8, \cdots, 8)$ is automatically very ample [6, pp. 83-84], the naive height is defined. A property of the Néron-Tate heights shows

$$q_A(\mathscr{M}, P) = q_A(f^*\mathscr{N}, P) = q_B(\mathscr{N}, f(P)) \quad \text{for } P \in A(F).$$

Thus we have

$$\begin{aligned} &\# \{P \in A(F) \mid q_A(\mathscr{M}, P) < \frac{1}{C\Delta D}\} \\ &\leq \deg f \cdot \# \{Q \in B(F) \mid q_B(\mathscr{N}, Q) < \frac{1}{C\Delta D}\} \\ &= \deg \mathscr{L} \cdot \# \{Q \in B(F) \mid q_B(\mathscr{N}, Q) < \frac{1}{C\Delta D}\}. \end{aligned}$$

By virtue of the Masser-David theorem, we obtain

$$\begin{aligned} &\# \{P \in A(F) \mid q_A(\mathscr{M}, P) < \frac{1}{C\Delta D}\} \\ &< \deg \mathscr{L} \cdot C \cdot h_{\mathrm{n}}(B, \mathscr{N})^{3g/2} \Delta^{3g/2} (1 + \log \Delta)^g D^g (1 + \log D)^g \\ &\leq \deg \mathscr{L} \cdot C \cdot h_{\mathrm{n}}(B, \mathscr{N}, t)^{3g/2} \Delta^{3g/2} (1 + \log \Delta)^g D^g (1 + \log D)^g. \end{aligned}$$

Together with the additive law of Néron-Tate heights, we get

$$\begin{aligned} q_A(\mathscr{M}, P) &= q_A((\mathscr{L} \otimes (-1)^*\mathscr{L})^{\otimes 4}, P) \\ &= 4 \cdot q_A(\mathscr{L}, P) + 4 \cdot q_A((-1)^*\mathscr{L}, P) \\ &= 4 \cdot q_A(\mathscr{L}, P) + 4 \cdot q_A(\mathscr{L}, -P) \\ &= 8 \cdot q_A(\mathscr{L}, P). \end{aligned}$$

Theorem 0.1 gives the desired inequality. □

**Proof of Theorem 0.4.** Notation being as above, for $P \in A(F)$ we have

$$q_A(\mathscr{L}, P) = \frac{1}{8} q_B(\mathscr{N}, f(P)).$$

By Theorem 0.1, it suffices to prove the theorem for $(B, \mathscr{N})$. The conclusion is immediate by an easy argument using the theorem of Masser and David. □

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