Stationary Solutions of the Heat Convection Equations in Exterior Domains

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1. Introduction. Let $\Omega = K^c \subset \mathbf{R}^3$ where K is a compact set whose boundary ∂K is of class C^2. We put $\partial\Omega = \Gamma = \partial K$. Then we consider the stationary problem for the heat convection equation (HCE) in Ω :

$$(1)\quad \begin{cases} (u\cdot\nabla)u = -(\nabla p)/\rho \\ \qquad + \{1-\alpha(\theta-\Theta_0)\}g + \nu\Delta u & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ (u\cdot\nabla)\theta = \kappa\Delta\theta & \text{in } \Omega, \end{cases}$$

$$(2)\quad u|_\Gamma = 0,\quad \theta|_\Gamma = \Theta_0 > 0,\quad \lim_{|x|\to\infty} u(x) = 0,\quad \lim_{|x|\to\infty}\theta(x) = 0,$$

where $u = u(x)$ is the velocity vector, $p = p(x)$ is the pressure and $\theta = \theta(x)$ is the temperature; $\nu, \kappa, \alpha, \rho$, and $g = g(x)$ are the kinematic viscosity, the thermal conductivity, the coefficient of volume expansion, the density at $\theta = \Theta_0$ and the gravitational vector, respectively.

As concerns the exterior problem of (HCE), Hishida [2] proved the global existence of the strong solution of the initial value problem (IVP) in the case that K is a ball. Recently, Ōeda-Matsuda [10] showed the existence and uniqueness of weak solutions of (IVP) when K is a compact set with the boundary of class C^2. In [10], the approach to prove the existence of weak solutions was "the extending domain method", that is, the exterior domain Ω was approximated by interior domains $\Omega_n = B_n \cap \Omega$ (B_n is a ball with radius n and center at O) as $n \to \infty$ (see Ladyzhenskaya [3]). On the other hand, Morimoto [6], [7] studied the stationary problem of (HCE) in interior domains and showed the existence and uniqueness of weak solutions. The purpose of the present paper is to show the existence of stationary weak solutions of (HCE) by using "the extending domain method". Moreover, we also study the uniqueness of a weak solution.

2. Preliminaries. We make several assumptions (A1)~(A3):

(A1) $\omega_0 \subset \operatorname{int} K$ (ω_0 is a neighbourhood of the origine O) and $K \subset B = B(O, d)$ which is a ball with radius d and center at O. (A2) $\partial\Omega = \Gamma = \partial K \in C^2$. (A3) $g(x)$ is a bounded and continuous vector function in $\mathbf{R}^3 \setminus \omega_0$. Moreover there exist $R_0 > 0$, $C_{R_0} > 0$ such that $|g| \le C_{R_0}/|x|^{\frac{5}{2}+\varepsilon}$ for $|x| \ge R_0$ ($\varepsilon > 0$ is arbitrary).

Remark 1. By (A3), we can take $C_w > 0$ such that $|g|\cdot|x|^{\frac{5}{2}+\varepsilon} \le C_w$ for $x \in \mathbf{R}^3\setminus\omega_0$. Moreover $g \in L^p(\Omega)$ for $p \ge \frac{6}{5}$.

Here, in order to transform the boundary condition on θ to a homogenuous one, we introduce an auxiliary function $\bar\theta$ (see [1] p.131, [11] p.175):

Lemma 2.1. *There exists a function* $\bar\theta$ *which satisfies the following properties* (i) ~ (iii): (i) $\bar\theta(\Gamma) = \Theta_0$. (ii) $\bar\theta \in C_0^2(\mathbf{R}^3)$. (iii) *For any* $\varepsilon > 0$ *and* $p \ge 1$, *we can retake* $\bar\theta$, *if necessary, such that* $\|\bar\theta\|_{L^p} < \varepsilon$.

Now we make a change of variable: $\theta = \hat\theta + \bar\theta$. And after changing of variable, we use the same letter θ. The system of equations (1) and (2) is transformed to the following:

$$(3)\quad \begin{cases} (u\cdot\nabla)u = -(\nabla p)/\rho - \alpha\theta g + \nu\Delta u \\ \qquad + \{1-\alpha(\bar\theta-\Theta_0)\}g & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ (u\cdot\nabla)\theta = \kappa\Delta\theta - (u\cdot\nabla)\bar\theta + \kappa\Delta\bar\theta & \text{in } \Omega, \end{cases}$$

$$(4)\quad u|_\Gamma = 0,\quad \theta|_\Gamma = 0,\quad \lim_{|x|\to\infty} u(x) = 0,\quad \lim_{|x|\to\infty}\theta(x) = 0.$$

We use several function spaces. G denotes Ω or Ω_n.

$W^{k,p}(G) = \{u\,;\, D^\alpha u \in L^p(G), |\alpha| \le k\}$, $W_0^{k,p}(G)$ = the completion of $C_0^k(G)$ in $W^{k,p}(G)$, $D_\sigma(G) = \{\varphi \in C_0^\infty(G)\,;\, \operatorname{div}\varphi = 0\}$, $D(G) = \{\psi \in C_0^\infty(G\cup\Gamma)\,;\, \psi(\Gamma) = 0\}$, $H_\sigma(G)$ (resp. $H_\sigma^1(G)$) = the completion of $D_\sigma(G)$ in $L^2(G)$ (resp. $W^1(G)$), V (resp. W) = the completion of $D_\sigma(\Omega)$ (resp. $D(\Omega)$) in $\|\cdot\|_{N(\Omega)}$, where $\|u\|_{N(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$, $H_0^1(\Omega_n)$ = the completion of $D(\Omega_n)$ in $W^{1,2}(\Omega_n)$ (it turns out $H_0^1(\Omega_n) = W_0^{1,2}(\Omega_n)$).

We make use of some inequalities. Constants which appear in those inequalities depend only on the dimension and they are independent of domains (see Chap. I of [3]).

Lemma 2.2. *Suppose the space dimension is* 3 *and* G *is bounded or unbounded. Then*

(i) *For* $u \in W_0^{1,2}(G)$ (*or* V *or* W), *we have*

(5) $\| u \|_{L^6(G)} \le c \| \nabla u \|_{L^2(G)}$, *where* $c = (48)^{1/6}$.

(ii) (Hölder's inequality). *If each integral makes sense, then we have*

(6) $$| ((u \cdot \nabla) v, w)_G | \le 3^{\frac{1}{p}+\frac{1}{r}} \| u \|_{L^p(G)} \cdot \| \nabla v \|_{L^p(G)} \cdot \| w \|_{L^r(G)},$$

where $p, q, r > 0$ *and* $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 1$.

3. Results. We will give the definition of a weak solution.

Definition 3.1. ${}^t(u, \theta) \in V \times W$ *is called a stationary weak solution of* (HCE) *if it satisfies* (7) *and* (8) *for all* $\varphi \in D_\sigma(\Omega)$ *and* $\phi \in D(\Omega)$:

(7) $$((u \cdot \nabla)\varphi, u) - \nu(\nabla u, \nabla\varphi) - (\alpha g\theta, \varphi) + (\{1 - \alpha(\bar\theta - \Theta_0)\}g, \varphi) = 0,$$

(8) $$((u \cdot \nabla)\phi, \theta) - \kappa(\nabla\theta, \nabla\phi) - ((u \cdot \nabla)\bar\theta, \phi) - \kappa(\nabla\bar\theta, \nabla\phi) = 0.$$

Remark 2. If $u \in V$, $\theta \in W$, then $u(\Gamma) = 0$, $\theta(\Gamma) = 0$, and moreover by (5) $\lim_{|x| \to \infty} u(x) = 0$, $\lim_{|x| \to \infty} \theta(x) = 0$.

Then we have following results.

Theorem 3.2. *Suppose assumpions* (A1), (A2), *and* (A3) *are satisfied. Then a stationary weak solution of* (HCE) *exists.*

Theorem 3.3. *Let assumpions* (A1), (A2), *and* (A3) *be satisfied. If there exists a stationary weak solution satisfying the following Condition* (C):

$$3c\left(\frac{1}{\nu} \| u \|_{L^3(\Omega)} + \frac{3c^2\alpha \| g \|_{L^{\frac{3}{2}}(\Omega)}}{\kappa\nu} \| \theta \|_{L^3(\Omega)}\right) < 1$$

(*where* $c = (48)^{1/6}$),

then the weak solution is unique.

4. Proof of results. According to the approach of "the extending domain method", we first present a lemma which ensures the existence of weak solutions of interior problems (P_n) in domains $\Omega_n = B_n \cap \Omega$. The interior problem ($P_n$) is as follows:

(9) $$\begin{cases} (v \cdot \nabla) v = -(\nabla p)/\rho - \alpha\Theta g + \nu\Delta v \\ \qquad\qquad + \{1 - \alpha(\bar\theta - \Theta_0)\}g & \text{in } \Omega_n, \\ \operatorname{div} v = 0 & \text{in } \Omega_n, \\ (v \cdot \nabla)\Theta = \kappa\Delta\Theta - (v \cdot \nabla)\bar\theta + \kappa\Delta\bar\theta & \text{in } \Omega_n, \end{cases}$$

(10) $v|_{\partial\Omega_n} = 0$, $\Theta|_{\partial\Omega_n} = 0$, where $\partial\Omega_n = \Gamma + \partial B_n$.

Here we give the definition of a weak solution for the problem (P_n):

Definition 4.1. ${}^t(v, \Theta) \in H_\sigma^1(\Omega_n) \times H_0^1(\Omega_n)$ *is called a weak solution for* (P_n) *if it satisfies the following*:

(11) $$((v \cdot \nabla)\varphi, v) - \nu(\nabla v, \nabla\varphi) - (\alpha g\Theta, \varphi) + (\{1 - \alpha(\bar\theta - \Theta_0)\}g, \varphi) = 0, \text{ for } \varphi \in D_\sigma(\Omega_n),$$

(12) $$((v \cdot \nabla)\phi, \Theta) - \kappa(\nabla\Theta, \nabla\phi) - ((v \cdot \nabla)\bar\theta, \phi) - \kappa(\nabla\bar\theta, \nabla\phi) = 0, \text{ for } \phi \in D(\Omega_n).$$

Now we will state a key lemma to carry out "the extending domain method".

Lemma 4.2. *Let assumptions* (A1), (A2), *and* (A3) *be satisfied. Then we can choose an appropriate extension* $\bar\theta$ *which is independent of* Ω_n *such that, making use of it in common to all* Ω_n, *we can construct a weak solution* ${}^t(v_n, \Theta_n)$ *of* (P_n).

Proof of Lemma 4.2. We use Galerkin's method and Brouwer's fixed point theorem. Let n be arbitrarily fixed. Let $\{\varphi_j\} \subset D_\sigma(\Omega_n)$ (resp. $\{\phi_j\} \subset D(\Omega_n)$) be a sequence of functions, linearly independent and total in $H_\sigma^1(\Omega_n)$ (resp. $H_0^1(\Omega_n)$). Since Ω_n is bounded, we can take them such that $(\nabla\varphi_j, \nabla\varphi_k) = \delta_{jk}$, $(\nabla\phi_j, \nabla\phi_k) = \delta_{jk}$. We put

$$v^{(m)} = \sum_{j=1}^m \xi_j\varphi_j, \quad \Theta^{(m)} = \sum_{j=1}^m \eta_j\phi_j,$$

then we consider the next system of equations:

(13) $$((v^{(m)} \cdot \nabla)\varphi_j, v^{(m)}) - \nu(\nabla v^{(m)}, \nabla\varphi_j) - (\alpha g\Theta^{(m)}, \varphi_j) + (\{1 - \alpha(\bar\theta - \Theta_0)\}g, \varphi_j) = 0,$$

(14) $$((v^{(m)} \cdot \nabla)\phi_j, \Theta^{(m)}) - \kappa(\nabla\Theta^{(m)}, \nabla\phi_j) - ((v^{(m)} \cdot \nabla)\bar\theta, \phi_j) - \kappa(\nabla\bar\theta, \nabla\phi_j) = 0,$$

where $1 \le j \le m$. Using the representations of $v^{(m)}$, $\Theta^{(m)}$, we have

(15) $$\sum_{k,l} \xi_k\xi_l((\varphi_k \cdot \nabla)\varphi_j, \varphi_l) - \nu\xi_j - \sum_k \eta_k(\alpha g\phi_k, \varphi_j) + (\{1 - \alpha(\bar\theta - \Theta_0)\}g, \varphi_j) = 0,$$

(16) $$\sum_{k,l} \xi_k\eta_l((\varphi_k \cdot \nabla)\phi_j, \phi_l) - \kappa\eta_j - \sum_k \xi_k((\varphi_k, \nabla)\bar\theta, \phi_j) - \kappa(\nabla\bar\theta, \nabla\phi_j) = 0,$$

where $1 \le j \le m$.

We put $(\xi ; \eta) = (\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_m)$, $P(\xi ; \eta) = (P_1(\xi ; \eta), \cdots, P_{2m}(\xi ; \eta))$ and

(17) $$P_j(\xi ; \eta) \equiv \frac{1}{\nu}\{\sum_{k,l} \xi_k\xi_l((\varphi_k \cdot \nabla)\varphi_j, \varphi_l) - \sum_k \eta_k(\alpha g\phi_k, \varphi_j) + (\{1 - \alpha(\bar\theta - \Theta_0)\}g, \varphi_j)\},$$

(18) $$P_{m+j}(\xi ; \eta) \equiv \frac{1}{\kappa}\{\sum_{k,l} \xi_k\eta_l((\varphi_k \cdot \nabla)\phi_j, \phi_l) - \sum_k \xi_k((\varphi_k, \nabla)\bar\theta, \phi_j) - \kappa(\nabla\bar\theta, \nabla\phi_j)\},$$

where $1 \le j \le m$. Then our problem is reduced to obtain a fixed point of $P : \boldsymbol{R}^{2m} \to \boldsymbol{R}^{2m}$. Now we will use Brouwer's fixed point theorem.

Namely, if all possible solutions $(\xi\,;\eta)$ of the equation $(\xi\,;\eta)=\lambda P(\xi\,;\eta)$ for $\lambda\in[0,1]$ stay in a some ball $\|(\xi\,;\eta)\|\le r$, then there exists a fixed point of P. Multiplying (15)(resp. (16)) by ξ_j (resp. η_j), summing up with respect to j and noting $((v^{(m)}\cdot\nabla)v^{(m)},v^{(m)})=0$, $((v^{(m)}\cdot\nabla)\Theta^{(m)},\Theta^{(m)})=0$, we have:

$$(19)\qquad \nu(\nabla v^{(m)},\nabla v^{(m)})+(ag\Theta^{(m)},v^{(m)})-(\{1-\alpha(\bar\theta-\Theta_0)\}g,v^{(m)})=0,$$

$$(20)\quad \kappa(\nabla\Theta^{(m)},\nabla\Theta^{(m)})+((v^{(m)}\cdot\nabla)\bar\theta,\Theta^{(m)})+\kappa(\nabla\bar\theta,\nabla\Theta^{(m)})=0.$$

Using the assumption (A3) and Lemma 2.2, we have from (19)

$$(21)\quad \nu\sum_{j=1}^m|\xi_j|^2=\nu\|\nabla v^{(m)}\|^2=\nu\lambda\sum_{j=1}^m P_j(\xi\,;\eta)\xi_j$$
$$\le\lambda\{|(\alpha g\Theta^{(m)},v^{(m)})|+(1+\alpha\Theta_0)|(g,v^{(m)})|+|(\alpha g\bar\theta,v^{(m)})|\}$$
$$\le\lambda\{3\alpha\|g\|_{\frac32}\cdot\|\Theta^{(m)}\|_6\cdot\|v^{(m)}\|_6+(1+\alpha\Theta_0)\|g\|_{\frac65}\cdot\|v^{(m)}\|_6+3\alpha\|g\|_{L^2(\Omega)}\cdot\|\bar\theta\|_3\cdot\|v^{(m)}\|_6\}$$
$$\le\lambda\{3c^2\alpha\|g\|_{\frac32}\cdot\|\nabla\Theta^{(m)}\|\cdot\|\nabla v^{(m)}\|+c(1+\alpha\Theta_0)\|g\|_{\frac65}\|\nabla v^{(m)}\|+3c\alpha\|g\|_{L^2(\Omega)}\cdot\|\bar\theta\|_3\cdot\|\nabla v^{(m)}\|\},$$

here $\|\cdot\|_p=\|\cdot\|_{L^p(\Omega_n)}$, $\|g\|_p=\|g\|_{L^p(\Omega)}$, $\|\bar\theta\|_p=\|\bar\theta\|_{L^p(\Omega)}$, $c=(48)^{1/6}$. From (21)

$$(22)\qquad \nu\|\nabla v^{(m)}\|\le\lambda\{3c^2\alpha\|g\|_{\frac32}\|\nabla\Theta^{(m)}\|+c(1+\alpha\Theta_0)\|g\|_{\frac65}+3c\alpha\|g\|_{L^2(\Omega)}\cdot\|\bar\theta\|_3\}.$$

Moreover we have from (20)

$$(23)\quad \kappa\sum_{j=1}^m|\eta_j|^2=\kappa\|\nabla\Theta^{(m)}\|^2$$
$$\le\lambda\{|((v^{(m)}\cdot\nabla)\Theta^{(m)},\bar\theta)|+\kappa|(\nabla\bar\theta,\nabla\Theta^{(m)})|\}$$
$$\le\lambda\{3c\|\nabla v^{(m)}\|\cdot\|\nabla\Theta^{(m)}\|\cdot\|\bar\theta\|_3+\kappa\|\nabla\bar\theta\|\cdot\|\nabla\Theta^{(m)}\|\},$$

from which we get

$$(24)\ \kappa\|\nabla\Theta^{(m)}\|\le\lambda\{3c\|\nabla v^{(m)}\|\cdot\|\bar\theta\|_3+\kappa\|\nabla\bar\theta\|\}.$$

Combining (22) and (24), then we have

$$(25)\ \nu\|\nabla v^{(m)}\|\le\lambda\{3c^2\alpha\|g\|_{\frac32}\lambda\kappa^{-1}(3c\|\nabla v^{(m)}\|\cdot\|\bar\theta\|_3+\kappa\|\nabla\bar\theta\|)+c(1+\alpha\Theta_0)\|g\|_{\frac65}+3c\alpha\|g\|_{L^2(\Omega)}\cdot\|\bar\theta\|_3\}.$$

Recalling (iii) of Lemma 2.1, we can take $\bar\theta$ satisfying

$$(26)\qquad \gamma\equiv\frac{9c^3\alpha\|g\|_{\frac32}}{\kappa\nu}\|\bar\theta\|_3<1.$$

We note $\bar\theta$ is taken in common not only in m but also for all $\Omega_n(n\ge1)$. Now we have for such $\bar\theta$

$$(27)\qquad \|\nabla v^{(m)}\|\le\frac{\lambda}{(1-\lambda^2\gamma)\nu}\{3c^2\alpha\|g\|_{\frac32}\lambda\|\nabla\bar\theta\|+c(1+\alpha\Theta_0)\|g\|_{\frac65}+3c\alpha\|g\|_{L^2(\Omega)}\cdot\|\bar\theta\|_3\}$$
$$=\frac{\lambda}{(1-\lambda^2\gamma)\nu}\{3c^2\alpha\|g\|_{\frac32}\lambda\|\nabla\bar\theta\|+c(1+\alpha\Theta_0)\|g\|_{\frac65}+\frac{\gamma\kappa\nu}{3c^2\|g\|_{\frac32}}\|g\|_{L^2(\Omega)}\}.$$

Combining (24) and (27), we find

$$(28)\ \|\nabla\Theta^{(m)}\|\le\frac{3c\|\bar\theta\|_3\lambda^2}{(1-\lambda^2\gamma)\kappa\nu}\Big\{3c^2\alpha\|g\|_{\frac32}\lambda\|\nabla\bar\theta\|+c(1+\alpha\Theta_0)\|g\|_{\frac65}+\frac{\gamma\kappa\nu}{3c^2\|g\|_{\frac32}}\|g\|_{L^2(\Omega)}\Big\}+\lambda\|\nabla\bar\theta\|$$
$$=\frac{\lambda}{1-\lambda^2\gamma}\|\nabla\bar\theta\|+\frac{\lambda^2\gamma}{1-\lambda^2\gamma}\Big\{\frac{1}{3c\alpha\|g\|_{\frac32}}(1+\alpha\Theta_0)\|g\|_{\frac65}+\frac{\gamma\kappa\nu}{9c^4\alpha\|g\|^2_{\frac32}}\|g\|_{L^2(\Omega)}\Big\}.$$

Since $0\le\lambda\le1$ and $\frac{1}{1-\lambda^2\gamma}\le\frac{1}{1-\gamma}$, we have from (27) and (28)

$$(29)\qquad \|\nabla v^{(m)}\|\le\frac{1}{(1-\gamma)\nu}\Big\{3c^2\alpha\|g\|_{\frac32}\|\nabla\bar\theta\|+c(1+\alpha\Theta_0)\|g\|_{\frac65}+\frac{\gamma\kappa\nu}{3c^2\|g\|_{\frac32}}\|g\|_{L^2(\Omega)}\Big\}\equiv r_1,$$

$$(30)\ \|\nabla\Theta^{(m)}\|\le\frac{1}{1-\gamma}\|\nabla\bar\theta\|+\frac{\gamma}{1-\gamma}\Big\{\frac{1}{3c\alpha\|g\|_{\frac32}}(1+\alpha\Theta_0)\|g\|_{\frac65}+\frac{\gamma\kappa\nu}{9c^4\alpha\|g\|^2_{\frac32}}\|g\|_{L^2(\Omega)}\Big\}\equiv r_2.$$

Thus we have gotten uniform estimates on $v^{(m)}$ and $\Theta^{(m)}$. Indeed, r_1 and r_2 are both independent of λ, m, n. Hence solutions of $(\xi\,;\eta)=\lambda P(\xi\,;\eta)$ for $\lambda\in[0,1]$ lie in a $\boldsymbol{R}^{2m}$-ball $\{\sum_{j=1}^m(|\xi_j|^2+|\eta_j|^2)\le r_1^2+r_2^2\equiv r^2\}$. Therefore, due to Brouwer's fixed point theorem, we have obtained a solution ${}^t(v^{(m)},\Theta^{(m)})$ of the equations (13) and (14) with the property (after getting the fixed point, repeat the same calculation as $\lambda=1$)

$$(31)\qquad \|\nabla v^{(m)}\|\le r_1,\quad \|\nabla\Theta^{(m)}\|\le r_2.$$

Then, thanks to (31), we can find subsequences $v^{(m)}$, $\Theta^{(m)}$ (we used the same letters) and $v\in H^1_\sigma(\Omega_n)$, $\Theta\in H^1_0(\Omega_n)$ such that

$v^{(m)}\to v$ weakly in $H^1_\sigma(\Omega_n)$, strongly in $H_\sigma(\Omega_n)$,
$\Theta^{(m)}\to\Theta$ weakly in $H^1_0(\Omega_n)$, strongly in $L^2(\Omega_n)$.

Passing to the limit in (13) and (14) as $m\to\infty$, we find that ${}^t(v,\Theta)$ is a desired weak solution. We skip the remaining part of the proof of Lemma 4.2.

Moreover, we state a lemma which we will use to prove Theorem 3.2.

Lemma 4.3. *Let* ${}^t(v_n,\Theta_n)$ *be a weak solution for* (P_n) *obtained in Lemma* 4.2. *Put* $u_n(x)=v_n(x)$ *if* $x\in\Omega_n$ *and* $u_n(x)=0$ *if* $x\in\Omega\setminus\Omega_n$;

$\theta_n(x) = \Theta_n(x)$ if $x \in \Omega_n$ and $\theta_n(x) = 0$ if $x \in \Omega \setminus \Omega_n$. Then it holds that ${}^t(u_n, \theta_n) \in V \times W$ and furthereore

$$\|\nabla u_n\| \le r_1, \ \|\nabla \theta_n\| \le r_2, \tag{32}$$

where r_1, r_2 be taken uniformly in n.

Proof of Lemma 4.3. It is easy to show ${}^t(u_n, \theta_n) \in V \times W$. As for the uniform estimate (32), by means of the lower semicontinuity of the norm of Hilbert space with respect to weak convergence, we have from (31) that $\|\nabla v_n\| \le r_1$ and $\|\nabla \Theta_n\| \le r_2$ (uniformly in n). But we can get these uniform estimates directly. Indeed, since $H^1_\sigma(\Omega_n)$ (resp. $H^1_0(\Omega_n)$) is a completion of $D_\sigma(\Omega_n)$ (resp. $D(\Omega_n)$) in $W^{1,2}(\Omega_n)$, from the weak form (11) and (12), we have

$$\nu(\nabla v_n, \nabla v_n) + (\alpha g \Theta_n, v_n) - (\{1 - \alpha(\bar\theta - \Theta_0)\}g, v_n) = 0, \tag{33}$$

$$\kappa(\nabla \Theta_n, \nabla \Theta_n) + ((v_n \cdot \nabla)\bar\theta, \Theta_n) + \kappa(\nabla\bar\theta, \nabla\Theta_n) = 0. \tag{34}$$

Then uniform estimates on ∇v_n, $\nabla \Theta_n$ follow from (33) and (34) by the similar calculation used in the proof of Lemma 4.2. Estimates (32) are immediate consequences of those on ∇v_n and $\nabla \Theta_n$.

Proof of Theorem 3.2. Considering uniform estimates $\|\nabla u_n\| \le r_1$ and $\|\nabla \theta_n\| \le r_2$ (uniform in n) in Lemma 4.3, applying Rellich's theorem and using the diagonal argument, we can choose subsequences $u_{n'}$, $\theta_{n'}$ and $u \in V$, $\theta \in W$ such that

(35) $u_{n'} \to u$ weakly in V, strongly in $L^2_{loc}(\Omega)$,

(36) $\theta_{n'} \to \theta$ weakly in W, strongly in $L^2_{loc}(\Omega)$.

Once we get such subsequences and limits, then we can show that ${}^t(u, \theta)$ becomes a stationary weak solution of (HCE). In fact, let ${}^t(\varphi, \psi)$ be an arbitrarily given test function, then we find a bounded domain Ω' and a number n_0 such that $\operatorname{supp}\varphi$, $\operatorname{supp}\psi \subset \Omega'$ and $\Omega' \subset \Omega_{n_0} \subset \Omega_n$ for all $n \ge n_0$. Then we have by Lemma 2.2

$$\begin{aligned}
&|((u_{n'} \cdot \nabla)\varphi, u_{n'})_\Omega - ((u \cdot \nabla)\varphi, u)_\Omega| \\
&\le |((u_{n'} \cdot \nabla)\varphi, u_{n'} - u)_{\Omega'}| + |(((u_{n'} - u) \cdot \nabla)\varphi, u)_{\Omega'}| \\
&\le 3\|u_{n'} - u\|_{L^2(\Omega')}\|u_{n'}\|_{L^6(\Omega)}\|\nabla\varphi\|_{L^3(\Omega')} + 3\|u\|_{L^6(\Omega)}\|u_{n'} - u\|_{L^2(\Omega')}\|\nabla\varphi\|_{L^3(\Omega')} \\
&\le 3\|u_{n'} - u\|_{L^2(\Omega')} c\|\nabla u_{n'}\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^3(\Omega')} + 3c\|\nabla u\|_{L^2(\Omega)}\|u_{n'} - u\|_{L^2(\Omega')}\|\nabla\varphi\|_{L^3(\Omega')} \\
&\le 3c \cdot (r_1 + \|\nabla u\|_{L^2(\Omega)})\|\nabla\varphi\|_{L^3(\Omega')} \cdot \|u_{n'} - u\|_{L^2(\Omega')},
\end{aligned} \tag{37}$$

and this implies the right hand side of (37) goes to 0 as $n' \to \infty$. Similarly

$$\begin{aligned}
&|((u_{n'} \cdot \nabla)\psi, \theta_{n'})_\Omega - ((u \cdot \nabla)\psi, \theta)_\Omega| \\
&\le 3\|\theta_{n'} - \theta\|_{L^2(\Omega')}\|u_{n'}\|_{L^6(\Omega)}\|\nabla\psi\|_{L^3(\Omega')} + 3\|\theta\|_{L^6(\Omega)}\|u_{n'} - u\|_{L^2(\Omega')}\|\nabla\psi\|_{L^3(\Omega')} \\
&\le 3c \cdot \|\nabla\psi\|_{L^3(\Omega')}(r_1\|\theta_{n'} - \theta\|_{L^2(\Omega')} + \|\nabla\theta\|_{L^2(\Omega)} \cdot \|u_{n'} - u\|_{L^2(\Omega')}),
\end{aligned} \tag{38}$$

hence we see the right hand side of (38) tends to 0 as $n' \to \infty$. We find the other terms in the weak formula also converge to the corresponding ones. Thus we see ${}^t(u, \theta)$ is a stationary weak solution of (HCE).

Now, we will return to the claim (35) and (36). Since $\|\nabla u_n\| \le r_1$, we can select a subsequence u'_n and $u \in V$ such that $u'_n \to u$ weakly in V. Moreover, put $K_j = \bar\Omega_j$, then we have a sequence of compact sets $\{K_j\}_{j=1}^\infty$ satisfying $K_1 \Subset K_2 \Subset \cdots \to \Omega$ $(j \to \infty)$. We note that for any compact set $F \subset \Omega$ there is a number j_0 such that $F \subset K_{j_0}$. Now for each K_j we choose $\alpha_j(x) \in C_0^\infty(\Omega)$ satisfying $0 \le \alpha_j \le 1$, $\alpha_j|_{K_j} \equiv 1$, and $(K_j \subset)$ $\operatorname{supp}\alpha_j \subset \Omega_{j+1}$. Here let us construct $\{u_{n'}\}$. First we make a sequence $\{\alpha_1(x)u_n(x)\}_{n=1}^\infty$, then this becomes a uniformly bounded sequence of $W_0^{1,2}(\Omega_2)$. In fact, since $u_n(\Gamma) = 0$, using Poincaré's inequality on Ω_2, we have $\|\alpha_1 u_n\|_{\Omega_2} \le \|u_n\|_{\Omega_2} \le \dfrac{d_2}{\sqrt{2}}\|\nabla u_n\|_{\Omega_2} \le \dfrac{d_2}{\sqrt{2}} r_1$ (for $n \ge 1$), where $\|\cdot\|_{\Omega_j} = \|\cdot\|_{L^2(\Omega_j)}$, $d_j =$ the diameter of Ω_j. Moreover

$$\begin{aligned}
\|\nabla(\alpha_1 u_n)\|_{\Omega_2} &\le \|(\nabla\alpha_1)u_n\|_{\Omega_2} + \|\alpha_1(\nabla u_n)\|_{\Omega_2} \\
&\le |||\nabla\alpha_1||| \cdot \|u_n\|_{\Omega_2} + |||\alpha_1||| \cdot \|\nabla u_n\|_{\Omega_2} \\
&\le \frac{d_2}{\sqrt{2}} |||\nabla\alpha_1|||\, r_1 + |||\alpha_1|||\, r_1,
\end{aligned}$$

where $|||w||| = \operatorname{ess.sup}_{x\in\Omega_2}|w(x)|$. By these estimates we see $\{\alpha_1 u_n\}$ is uniformly bounded in $W_0^{1,2}(\Omega_2)$. Hence by virture of Rellich's theorem, there is a subsequence $\{\alpha_1 u_{1p}\}_{p=1}^\infty$ such that it converges strongly in $L^2(\Omega_2)$, and consequently the sequence $\{u_{1p}\}_{p=1}^\infty$ is a strongly convergent one in $L^2(K_1)$. Next we consider a sequence $\{\alpha_2(x)u_{1p}(x)\}_{p=1}^\infty$. Then we see it consists of a uniformly bounded sequence in $W_0^{1,2}(\Omega_3)$, so we can select a suitable subsequence $\{\alpha_2 u_{2p}\}_{p=1}^\infty$ such that it converges strongly in $L^2(\Omega_3)$ and $\{u_{2p}\}_{p=1}^\infty$ coverges strongly in $L^2(K_2)$. We go on such a procedure. Choosing diagonal components and denoting them by $\{u_{n'}\}_{n'=1}^\infty$, then it converges on all K_j in $L^2(K_j)$ sense. As to $\{\theta_{n'}\}_{n'=1}^\infty$, we can show in a similar way. Thus we have shown the claims (35) and (36). Hence we have established

Theorem 3.2.

Proof of Theorem 3.3. Let ${}^t(u_i, \theta_i)\,(i = 1,2)$ be two stationary weak solutions of (HCE). Subtract corresponding weak fomulas. We put $u = u_1 - u_2$ and $\theta = \theta_1 - \theta_2$. Since $D_\sigma(\Omega)$ (resp. $D(\Omega)$) is dense in V (resp. W), we can replace $\varphi \in D_\sigma(\Omega)$ (resp. $\psi \in D(\Omega)$) by u (resp. θ). Using $((u_2 \cdot \nabla)u, u) = 0$, $((u_2 \cdot \nabla)\theta, \theta) = 0$, we obtain

(39) $\nu \| \nabla u \|^2 = ((u \cdot \nabla)u, u_1) - (\alpha g \theta, u)$,

(40) $\kappa \| \nabla \theta \|^2 = ((u \cdot \nabla)\theta, \theta_1) - ((u \cdot \nabla)\bar{\theta}, \theta)$.

In view of the assumption (A3) and Lemma 2.2, we have from (39)

(41) $$\begin{aligned}\nu \| \nabla u \|^2 &\le 3 \| u \|_6 \| \nabla u \| \cdot \| u_1 \|_3 + 3\alpha \| g \|_{\frac{3}{2}} \cdot \| \theta \|_6 \cdot \| u \|_6 \\ &\le 3c \| \nabla u \|^2 \cdot \| u_1 \|_3 + 3c^2\alpha \| g \|_{\frac{3}{2}} \| \nabla \theta \| \cdot \| \nabla u \|.\end{aligned}$$

If $\| \nabla u \| \neq 0$, then the above inequality implies

(42) $\nu \| \nabla u \| \le 3c \| \nabla u \| \cdot \| u_1 \|_3 + 3c^2 \alpha \| g \|_{\frac{3}{2}} \| \nabla \theta \|$.

On the other hand, we have by (40)

(43) $$\begin{aligned}\kappa \| \nabla \theta \|^2 &\le 3 \| u \|_6 \| \nabla \theta \| \cdot \| \theta_1 \|_3 + 3 \| u \|_6 \| \nabla \theta \| \cdot \| \bar{\theta} \|_3 \\ &\le 3c \| \nabla u \| \cdot \| \nabla \theta \| \cdot (\| \theta_1 \|_3 + \| \bar{\theta} \|_3).\end{aligned}$$

If $\| \nabla \theta \| \neq 0$, then we find

(44) $\kappa \| \nabla \theta \| \le 3c \| \nabla u \| (\| \theta_1 \|_3 + \| \bar{\theta} \|_3)$.

Substituting (44) into (42), we obtain

(45) $$\nu \| \nabla u \| \le 3c \| \nabla u \| \cdot \| u_1 \|_3 + 9c^3 \alpha \| g \|_{\frac{3}{2}} \kappa^{-1} \| \nabla u \| (\| \theta_1 \|_3 + \| \bar{\theta} \|_3).$$

Since we assumed $\| \nabla u \| \neq 0$, we have

(46) $$1 \le 3c \Big\{ \frac{1}{\nu} \| u_1 \|_3 + \frac{3c^2 \alpha \| g \|_{\frac{3}{2}}}{\kappa \nu} (\| \theta_1 \|_3 + \| \bar{\theta} \|_3) \Big\}.$$

However, we have taken $\bar{\theta}$ in (26) such that $\gamma = \dfrac{9c^3 \alpha \| g \|_{\frac{3}{2}}}{\kappa \nu} \| \bar{\theta} \|_3 < 1$, then (46) implies

(47) $$1 \le 3c \Big(\frac{1}{\nu} \| u_1 \|_3 + \frac{3c^2 \alpha \| g \|_{\frac{3}{2}}}{\kappa \nu} \| \theta_1 \|_3 \Big) + \gamma.$$

Since $\gamma (0 < \gamma < 1)$ can be taken arbitrarily, if $\| u_1 \|_3$, $\| \theta_1 \|_3$ satisfy Condition (C), then (47) leads us a contradiction. Hence it must be that $\| \nabla u \| = \| \nabla \theta \| = 0$. Therefore we find $u =$ const. and $\theta =$ const. But $u(\Gamma) = \theta(\Gamma) = 0$, hence $u = 0$ and $\theta = 0$. Thus we have proved the uniqueness theorem.

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