Regular and Stable Points in Dirichlet Problem

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Consider a subregion M of *Carathéodory type* of the extended Euclidean space $\bar{\boldsymbol{R}}^d = \boldsymbol{R}^d \cup \{\infty\}$ of dimension $d \geq 2$, i.e. M is a subregion of $\bar{\boldsymbol{R}}^d$ such that the boundary ∂M of M is contained in $\boldsymbol{R}^d$ and $\partial \bar{M} = \partial M$. A sequence $(M_i)_{i \geq 1}$ of subregions M_i of $\bar{\boldsymbol{R}}^d$ is referred to as a *squeezer* of $\bar{M}$ if $M_i \supset M_{i+1} \supset \bar{M}$ for every $i \geq 1$ and $\cap_{i \geq 1} M_i = \bar{M}$. For any $f \in C(\bar{\boldsymbol{R}}^d)$ we denote by H_f^M the harmonic Dirichlet solution for the boundary function $f \mid \partial M$ on M obtained by the Perron-Wiener-Brelot method (cf. e.g. [4]). It is known as the Wiener type theorem that the sequence $(H_f^{M_i})_{i \geq 1}$ converges pointwise on $\bar{M}$ and locally uniformly on M for any $f \in C(\bar{\boldsymbol{R}}^d)$ and for any squeezer $(M_i)_{i \geq 1}$ of $\bar{M}$. It is convenient to introduce the notation

$$H_f^{\bar{M}}(x) := \lim_{i \to \infty} H_f^{M_i}(x) \quad (x \in \bar{M})$$

which is harmonic on M and depends only on $f \mid \partial M$ and $\bar{M}$ independent of the choice of the squeezer $(M_i)_{i \geq 1}$. The function $H_f^{\bar{M}}$ is sometimes referred to as the external solution of the Dirichlet problem for the domain M with the boundary function f and also given by

$$H_f^{\bar{M}}(x) = \int_{\partial M} f(y)\, d\beta_{\bar{M}^c} \varepsilon_x(y), \tag{1}$$

where ε_x is the Dirac measure with its support at x and $\beta_{\bar{M}^c}$ denotes the balayage operation for the set $\bar{M}^c$ (cf. [6, §5 in Chap. V]). The Dirichlet problem is said to be *stable inside* M (*stable in* $\bar{M}$, resp.) if $H_f^{\bar{M}} = H_f^M$ on M (if $(H_f^{M_i})_{i \geq 1}$ converges uniformly to f on ∂M, resp.). The stability in $\bar{M}$ implies the stability inside M. In particular, the stability in $\bar{M}$ is closely related to the harmonic approximation question (cf. e.g. [6], [3], [1], etc.). To judge the stability for concrete regions it is convenient to localize the stability. In his celebrated paper [5] Keldysh introduced the following notion: a boundary point $y \in \partial M$ is said to be a *stable point* if $H_f^{\bar{M}}(y) = f(y)$ for every $f \in C(\bar{\boldsymbol{R}}^d)$. A point $y \in \partial M$ which is not a stable point is termed as an *unstable point*. In view of (1) it is readily seen that $y \in \partial M$ is a stable point if and only if y is a regular point of the set $\bar{M}^c$ in the sense of [6, Chap. V].

In terms of stable points Keldysh [5] showed the following: the Dirichlet problem is stable inside M if and only if the set of all unstable points in ∂M is of harmonic measure zero relative to M ; the Dirichlet problem is stable in $\bar{M}$ if and only if every boundary point in ∂M is stable. As for the relation of stability of boundary points to the regularity (cf. e.g. [4]) of them, Keldysh [5] proved that a stable boundary point $y \in \partial M$ is automatically a regular boundary point for the Dirichlet problem on M but there is an example (i.e. the so called Keldysh ball (cf. no. 12 below)) indicating that the converse of the above is not true. There are many handy geometric criterion for the regularity and therefore it will be usefull to give a practical geometric condition under which the regularity implies the stability for boundary points. The *purpose* of this paper is to give such an easily applicable condition. Roughly speaking (cf. no. 3 below for precise definition), a boundary point $y \in \partial M$ is said to be *graphic* if one of the following two conditions is satisfied: there exist a neighborhood U of y, a Cartesian coordinate $x = (x^1, \cdots, x^{d-1}, x^d) = (x', x^d)$, and a continuous function $\phi(x')$ of x' such that $(\partial M) \cap U$ is represented as the graph of the function $x^d = \phi(x')$ and $M \cap U$ is situated on only one side of the graph; there exist a neighborhood U of y, a polar coordinate (r, ξ) $(r \geq 0, |\xi| = 1)$, and a continuous function $\phi(\xi) \geq 0$ of ξ such that $(\partial M) \cap U$ is represented as the graph of the function $r = \phi(\xi)$ and $M \cap U$ is situated on only one side of the graph.

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If M has a C^1-boundary ∂M, or more generally, if M is a Lipschitz domain, then any boundary point of M is graphic (and regular). The vertex of the Lebesgue spine (cf. e.g. [4]) is an example of an irregular graphic point. We will prove the following result.

2. Theorem. *A graphic boundary point $y \in \partial M$ of a Carathéodory domain M is a stable boundary point of M if and only if y is a regular boundary point of M for the Dirichlet problem on M.*

As already stated $y \in \partial M$ is stable if and only if it is regular for $\bar{M}^c$. Hence the above result may be restated as follows: a graphic boundary point $y \in \partial M$ of a Carathéodory domain M is regular for M^c if and only if y is a regular point for $\bar{M}^c$. A probabilistic proof to the result in this restated form is given by R. Howard (see the recent paper [2] of Bass and Burdzy) for the case $y \in \partial M$ is a Cartesian graphic point. In contrast with this our proof is analytic and covers not only the case $y \in \partial M$ is a Cartesian graphic point but also the case $y \in \partial M$ is a polar graphic point as well (see no. 3 below). The proof of Theorem 2 will be given in nos. 8 and 10 below.

3. Graphic points. For a subregion Ω of $\bar{\boldsymbol{R}}^d = \boldsymbol{R}^d \cup \{\infty\}$ we denote by $\partial\Omega$ the boundary $(\bar{\Omega} \setminus \Omega) \cap \boldsymbol{R}^d$ of Ω relative to $\boldsymbol{R}^d$. We say that a point $y \in \partial\Omega$ is a *graphic point* for Ω if y is either a Cartesian graphic point or a polar graphic point for Ω in the sense described below. First, a point $y \in \partial\Omega$ is referred to as a *Cartesian graphic point* for Ω if there are a Cartesian coordinate $x = (x^1, \dots, x^{d-1}, x^d) = (x', x^d)$ in $\boldsymbol{R}^d$ for which the coordinate of y is the origin $0 = (0', 0)$, two positive constants $r_0 > 0$ and $s_0 > 0$, and a function $\phi \in C(B'(0, r_0))$ with $\phi(0) = 0$ and $|\phi| < s_0$ on $B'(0, r_0)$, where $B'(0, r_0)$ is the open disc in $\boldsymbol{R}^{d-1}$ with radius r_0 centered at the origin $0 (= 0')$ of $\boldsymbol{R}^{d-1}$, such that

$$\Omega \cap R_0 = \{(x', x^d) : \phi(x') < x^d < s_0 \ (x' \in B'(0, r_0))\}, \tag{4}$$

where $R_0 = B'(0, r_0) \times (-s_0, s_0)$, and

$$(\partial\Omega) \cap R_0 = \{(x', x^d) : x^d = \phi(x') \ (x' \in B'(0, r_0))\}. \tag{5}$$

We denote by $S^{d-1} = \{\xi \in \boldsymbol{R}^d : |\xi| = 1\}$ the unit sphere in $\boldsymbol{R}^d$. By $\beta(\xi, \delta)$ we mean the geodesic ball (spherical cap) on S^{d-1} with radius $\delta > 0$ centered at $\xi \in S^{d-1}$ so that $\beta(\xi, \delta) = \{\eta \in S^{d-1} : d_{S^{d-1}}(\xi, \eta) < \delta\}$, where $d_{S^{d-1}}$ denotes the natural Riemannian distance on S^{d-1}. Let the number D stand for the geodesic distance between the north and south poles of S^{d-1}. Then, a point $y \in \partial\Omega$ is referred to as a *polar graphic point* for Ω if there exist a polar coordinate (r, ξ) $(r \in [0, \infty),\ \xi \in S^{d-1})$ in $\boldsymbol{R}^d$ for which the polar coordinate of y is (ρ, η) or $y = \rho\eta\ (\rho > 0)$, two constants $\delta_0 \in (0, D/4)$ and $\tau_0 \in (0, \rho)$, and a function $\phi \in C(\beta(\eta, \delta_0))$ with $\phi(\eta) = \rho$ and $|\phi - \rho| < \tau_0$ on $\beta(\eta, \delta_0)$ such that

$$\Omega \cap R_0 = \{(r, \xi) : \rho - \tau_0 < r < \phi(\xi) \ (\xi \in \beta(\eta, \delta_0))\}, \tag{6}$$

where $R_0 = (\rho - \tau_0, \rho + \tau_0) \times \beta(\eta, \delta_0)$, and

$$(\partial\Omega) \cap R_0 = \{(r, \xi) : r = \phi(\xi) \ (\xi \in \beta(\eta, \delta_0))\}. \tag{7}$$

The function ϕ in (4) and (5) or in (6) and (7) is said to be the *local representing function* of Ω at $y \in \partial\Omega$. It is not difficult to construct an example of $y \in \partial\Omega$ which is a Cartesian (polar, resp.) graphic point for Ω but not a polar (Cartesian, resp.) graphic point for Ω. A region $\Omega \subset \bar{\boldsymbol{R}}^d$ with compact $\partial\Omega$ in $\boldsymbol{R}^d$ is referred to as a *continuous domain* (or a domain of type C) if every point $y \in \partial\Omega$ is a graphic point.

8. Proof of Theorem 2 (The case of Cartesian graphic point). Keldysh [5] proved that $y \in \partial M$ is a regular boundary point of M if y is a stable boundary point of M. Hence we only have to show that, under the assumption that $y \in \partial M$ is a graphic point for M, $y \in \partial M$ is a stable boundary point of M if y is a regular boundary point with respect to the harmonic Dirichlet problem on M. First we treat the case $y \in \partial M$ is a *Cartesian* graphic point. Hence there exist a Cartesian coordinate (x', x^d) in $\boldsymbol{R}^d$ for which the coordinate of y is the origin $0 = (0', 0)$ of $\boldsymbol{R}^d$, two constants $r_0 > 0$ and $s_0 > 0$, and a function $\phi \in C(B'(0, r_0))$ with $\phi(0) = 0$ and $|\phi| < s_0$ such that (4) and (5) are valid. Then we only have to show that $H_f^{\bar{M}}(0) = f(0)$, or equivalently, there is a squeezer $(M_i)_{i \ge 1}$ of $\bar{M}$ such that

$$\lim_{i \to \infty} H_f^{M_i}(0) = f(0) \tag{9}$$

for any $f \in C(\bar{\boldsymbol{R}}^d)$. For this purpose choose an arbitrary but then fixed number $\varepsilon > 0$. Then there exist two constants $r \in (0, r_0)$ and $s \in (0, s_0)$ such that $|\phi| < s/2$ on $B'(0, r)$, $|f(x) - f(0)| < \varepsilon$ for x in the closure of $R := B'(0, r) \times (-s, s)$, $M \cap R = \{(x', x^d) : \phi(x') < x^d <$

$s\ (x' \in B'(0, r))\}$, and $(\partial M \cap R) = \{(x', x^d) : x^d = \psi(x')\ (x' \in B'(0, r))\}$. Set $S = M \cap R$ and $S_i = S - e_i = \{x - e_i : x \in S\}$, where $e_i = (0', s/2i) \in \boldsymbol{R}^d (i = 1, 2, \ldots)$. We can find a squeezer $(\Omega_j)_{j\geq 1}$ of $\bar{M}$ such that each Ω_j is a polyhedron (i.e. a union of a finite number of intersections of a finite number of half spaces) and $\Omega_j \supset \bar{\Omega}_{j+1} (j = 1, 2, \ldots)$. There exists a subsequence $(j_i)_{i\geq 1}$ of $(j)_{j\geq 1}$ satisfying $\bar{\Omega}_{j_i} \cap R \subset S_i \cup S (i = 1, 2, \ldots)$. Observe that the upper side of S_i lies above ∂M. We now define a required squeezer $(M_i)_{i\geq 1}$ of $\bar{M}$ by $M_i = \Omega_{j_i} \cup S_i (i = 1, 2, \ldots)$. Consider two functions g and h on ∂S given as follows: $g(x) = f(0) + \varepsilon$ for $x \in (\partial M) \cap R$ and $g(x) = a + \varepsilon$ for $x \in \partial S \setminus (\partial M) \cap R$, where $a = \sup_{\bar{M}_1} |f|$; $h(x) = f(0) - \varepsilon$ for $x \in (\partial M) \cap R$ and $h(x) = -a - \varepsilon$ for $x \in \partial S \setminus (\partial M) \cap R$. Then define two functions g_i and h_i on ∂S_i by $g_i(x) = g(x + e_i)$ and $h_i(x) = h(x + e_i)$. It is easy to see that

$H^{S_i}_{g_i}(x) = H^S_g(x + e_i)$ and $H^{S_i}_{h_i}(x) = H^S_h(x + e_i)$

for every $x \in S_i\ (i = 1, 2, \ldots)$. Let p be any upper function for $\bar{H}^{S_i}_{g_i} = H^{S_i}_{g_i}$, i.e. p is lower bounded and hyperharmonic on S_i and $\lim\inf_{x\to z} p(x) \geq g_i(z)$ for every $z \in \partial S_i$. Then consider the function q on M_i defined by $q = \min\{p, a + \varepsilon\}$ on S_i and $q = a + \varepsilon$ on $M_i \setminus S_i$. Clearly q is an upper function for $\bar{H}^{M_i}_f = H^{M_i}_f$. Since $p \geq q \geq H^{M_i}_f$ on S_i, we have $H^{S_i}_{g_i} \geq H^{M_i}_f$ on S_i. Hence $H^S_g(e_i) = H^{S_i}_{g_i}(0) \geq H^{M_i}_f(0)$. As is easily seen by using the barrier criterion, an interior point in $(\partial S) \cap (\partial M)$ is a regular boundary point of S if and only if it is a regular boundary point of M (cf. e.g. [4]). Hence $0 \in \partial S$ is regular for S since $0 (= y) \in \partial M$ is regular for M. In view of $e_i \in S$ and $e_i \to 0\ (i \to \infty)$, we conclude that $\lim_{i\to\infty} H^S_g(e_i) = \lim_{x\in S, x\to 0} H^S_g(x) = g(0) = f(0) + \varepsilon$. Therefore we obtain

$$\lim_{i\to\infty} H^{M_i}_f(0) \leq f(0) + \varepsilon.$$

On repeating a similar argument by using h instead of g we can conclude that

$$\lim_{i\to\infty} H^{M_i}_f(0) \geq f(0) - \varepsilon.$$

From these two inequalities, by letting $\varepsilon \downarrow 0$, (9) follows. □

10. Proof of Theorem 2 (The case of polar graphic point). Next we show that, under the assumption that $y \in \partial M$ is a *polar* graphic point for M, $y \in \partial M$ is a stable boundary point of M if y is a regular boundary point of M. Then there exist a polar coordinate (r, ξ) for which y has the coordinate (ρ, η) or $y = \rho\eta (\rho > 0)$, two constants $\delta_0 \in (0, D/4)$ and $\tau_0 \in (0, \rho)$, and a function $\psi \in C(\beta(\eta, \delta_0))$ with $\psi(\eta) = \rho$ and $|\psi - \rho| < \tau_0$ such that (6) and (7) are valid. We are to prove that $H^M_f(\rho\eta) = f(\rho\eta)$, or equivalently, there is a squeezer $(M_i)_{i\geq 1}$ of $\bar{M}$ such that

$$\lim_{i\to\infty} H^{M_i}_f(\rho\eta) = f(\rho\eta) \tag{11}$$

for every $f \in C(\bar{\boldsymbol{R}}^d)$. For the purpose choose and then fix an arbitrary number $\varepsilon > 0$. Then there exist two constants $\tau \in (0, \tau_0)$ and $\delta \in (0, \delta_0)$ such that $|\psi - \rho| < \tau/2$ on $\beta(\eta, \delta)$, $|f(r\xi) - f(\rho\eta)| < \varepsilon$ for (r, ξ) in the closure of $R := (\rho - \tau, \rho + \tau) \times \beta(\eta, \delta)$, $M \cap R = \{r\xi : \rho - \tau < r < \psi(\xi)\ (\xi \in \beta(\eta, \delta))\}$, and $(\partial M) \cap R = \{r\xi : r = \psi(\xi)\ (\xi \in \beta(\eta, \delta))\}$. Let $S = M \cap R$ and $S_i = \lambda_i S = \{\lambda_i x : x \in S\}$, where $\lambda_i = 1 + \tau/2\rho i (i = 1, 2, \ldots)$. As in 8 let $(\Omega_j)_{j\geq 1}$ be a squeezer of $\bar{M}$ such that each Ω_j is a polyhedron and $\Omega_j \supset \bar{\Omega}_{j+1} (i = 1, 2, \ldots)$. There is a subseqence $(j_i)_{i\geq 1}$ of $(j)_{j\geq 1}$ such that $\bar{\Omega}_{j_i} \cap R \subset S_i \cup S\ (i = 1, 2, \ldots)$. Observe that the bottom of S_i lies below ∂M. We define a required squeezer $(M_i)_{i\geq 1}$ of $\bar{M}$ by $M_i = \Omega_{j_i} \cup S_i\ (i = 1, 2, \ldots)$. Consider two functions g and h on ∂S given as follows: $g(r\xi) = f(\rho\eta) + \varepsilon$ for $r\xi \in (\partial M) \cap R$ and $g(r\xi) = a + \varepsilon$ for $r\xi \in \partial S \setminus (\partial M) \cap R$, where $a = \sup_{\bar{M}_1} |f|$; $h(r\xi) = f(\rho\eta) - \varepsilon (r\xi \in (\partial M) \cap R)$ and $h(r\xi) = -a - \varepsilon$ on $\partial S \setminus (\partial M) \cap R$. Then take two functions g_i and h_i on ∂S_i given by $g_i(x) = g(\lambda_i^{-1} x)$ and $h_i(x) = h(\lambda_i^{-1} x)$ for $x \in \partial S_i (i = 1, 2, \ldots)$. It is easily seen that

$H^{S_i}_{g_i}(x) = H^S_g(\lambda_i^{-1} x)$ and $H^{S_i}_{h_i}(x) = H^S_h(\lambda_i^{-1} x)$

for every $x \in S_i (i = 1, 2, \ldots)$. As in 8, let p be any upper function for $\bar{H}^{S_i}_{g_i} = H^{S_i}_{g_i}$ on S_i. The function q defined on M_i by $\min\{p, a + \varepsilon\}$ on S_i and by $a + \varepsilon$ on $M_i \setminus S_i$ is an upper function for $\bar{H}^{M_i}_f = H^{M_i}_f$. From $p \geq q \geq H^{M_i}_f$ on S_i it follows that $H^{S_i}_{g_i} \geq H^{M_i}_f$ on S_i. Hence $H^S_g(\lambda_i^{-1}\rho\eta) = H^{S_i}_{g_i}(\rho\eta) \geq H^{M_i}_f(\rho\eta)\ (i = 1, 2, \ldots)$. Recall that $\rho\eta \in \partial S$ is regular for S because of $\rho\eta \in \partial M$ being regular for M. Since $\lambda_i^{-1}\rho\eta \in S$ and $\lambda_i^{-1}\rho\eta \to \rho\eta (i \to \infty)$, $\lim_{i\to\infty} H^S_g(\lambda_i^{-1}\rho\eta) = \lim_{r\xi\in S, r\xi\to\rho\eta} H^S_g(r\xi) = g(\rho\eta) = f(\rho\eta) + \varepsilon$ and a fortiori $\lim_{i\to\infty} H^{M_i}_f(\rho\eta) \leq f(\rho\eta) + \varepsilon$. Similarly $\lim_{i\to\infty} H^{M_i}_f(\rho\eta) \geq f(\rho\eta) - \varepsilon$. Letting $\varepsilon \downarrow 0$, we deduce (11). □

12. Corollary. *If M is a continuous domain, then the Dirichlet problem is stable inside M. If M is a continuous domain whose boundary points are all regular, then the Dirichlet problem is stable in $\bar{M}$.*

Proof. A continuous domain is of course a domain of Carathéodory type. Theorem 2 assures that the set of unstable boundary points of M coincides with the set of irregular boundary points of M, which is of capacity zero by the Kellogg theorem and a fortiori of harmonic measure zero relative to M. Thus the Keldysh theorem assures that the Dirichlet problem is stable inside M. The latter assertion also follows from the Keldysh theorem that the Dirichlet problem is stable in $\bar{M}$ if and only if the set of unstable boundary points of M is empty. □

13. Examples. We classify regions M of Carathéodory type into the following three types: a region M is of *type I* if the Dirichlet problem is stable in $\bar{M}$ and a fortiori it is stable inside M; *type II* if the Dirichlet problem is not stable in $\bar{M}$ but it is stable inside M; *type III* if the Dirichlet problem is not stable inside M and a fortiori it is not stable in $\bar{M}$. By the latter half of Corollary 12, Lipschitz domains M and convex domains M are typical examples of M of type I. A *Lebesgue ball* M is a topological ball (i.e. a domain M such that there exists a homeomorphism h of $\bar{M}$ onto the closed unit ball $B^d \cup S^{d-1}$ (where B^d is the open unit ball in $\boldsymbol{R}^d$) with $h(M) = B^d$ and $h(\partial M) = S^{d-1}$) such that ∂M is smooth except for a single point of ∂M which is the Lebesgue spine (cf. e.g. [4]). Then, by Corollary 12, we see that a Lebesgue ball M is of type II. Existence of a region M of type III is a nontrivial result of Keldysh [5]: the so called *Keldysh ball* is of type III, which is a topological ball M whose boundary ∂M consists of regular points and is of finite area such that the set of unstable points in ∂M is of zero area and of positive harmonic measure.

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