Connection Formulae for Solutions of a System of Partial Differential Equations Associated with the Confluent Hypergeometric Function Φ_2

By Shun SHIMOMURA

Department of Mathematics, Keio University

(Communicated by Kiyosi ITÔ, M. J. A., April 12, 1996)

1. Introduction. Consider the confluent hypergeometric function

(1) $\Phi_2(\beta, \beta', \gamma, x, y) = \sum_{m,n\geq 0} \frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}(1)_m(1)_n} x^m y^n$

convergent for $|x| < \infty$, $|y| < \infty$, in which $(\beta)_m = \Gamma(\beta + m)/\Gamma(\beta)$ (cf. [3]). This function satisfies a system of partial differential equations

(2) $$xz_{xx} + yz_{xy} + (\gamma - x)z_x - \beta z = 0,$$
$$yz_{yy} + xz_{xy} + (\gamma - y)z_y - \beta' z = 0,$$

which possesses the singular loci $x = 0$, $y = 0$, $x - y = 0$ of regular type and $x = \infty$, $y = \infty$ of irregular type. The solutions of system (2) constitute a three-dimensional vector space over $\boldsymbol{C}$. In what follows, we assume that none of the complex numbers β, β', $\gamma - \beta - \beta'$, $\beta - \gamma$, $\beta' - \gamma$, and $\beta + \beta'$ is an integer, and use the notation $e^{(\lambda)} = \exp(2\pi i\lambda)$.

It is known by Erdélyi [1,2] that, near the singular loci of irregular type, system (2) admits convergent solutions as follows:

$$u_0 = \Phi_2(\beta, \beta', \gamma, x, y) \quad (|x| < \infty, |y| < \infty),$$

$$v_1 = x^{\beta'-\gamma+1} y^{-\beta'} \Phi_1(\beta + \beta' - \gamma + 1, \beta', \beta' - \gamma + 2, x/y, x) \quad (|x| < |y|)$$
$$= x^{\beta'-\gamma+1}(y - x)^{-\beta'} \times \Phi_1(1 - \beta, \beta', \beta' - \gamma + 2, x/(x - y), -x) \quad (|x| < |x - y|),$$

$$v_2 = x^{-\beta} y^{\beta-\gamma+1} \times \Phi_1(\beta + \beta' - \gamma + 1, \beta, \beta - \gamma + 2, y/x, y) \quad (|y| < |x|),$$

$$v_3 = x^{\beta+\beta'-\gamma}(y - x)^{1-\beta-\beta'} e^x \Phi_1(1 - \beta, \gamma - \beta - \beta', 2 - \beta - \beta', (x - y)/x, y - x) \quad (|x - y| < |x|),$$

$$w_1 = y^{1-\gamma}\Gamma_1(\beta, \beta' - \gamma + 1, \gamma - 1, -x/y, -y) \quad (|x| < |y|)$$
$$= (y - x)^{1-\gamma} e^x \Gamma_1(\gamma - \beta - \beta', \beta' - \gamma + 1, \gamma - 1, x/(y - x), x - y) \quad (|x| < |x - y|),$$

$$w_2 = x^{1-\gamma}\Gamma_1(\beta', \beta - \gamma + 1, \gamma - 1, -y/x, -x) \quad (|y| < |x|),$$

$$w_3 = x^{1-\gamma} e^x \times \Gamma_1(\beta', 1 - \beta - \beta', \gamma - 1, (y - x)/x, x) \quad (|x - y| < |x|),$$

where

$$\Phi_1(\alpha, \beta, \gamma, x, y) = \sum_{m,n\geq 0} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}(1)_m(1)_n} x^m y^n,$$

$$\Gamma_1(\alpha, \beta, \beta', x, y) = \sum_{m,n\geq 0} \frac{(\alpha)_m(\beta)_{n-m}(\beta')_{m-n}}{(1)_m(1)_n} x^m y^n$$

are convergent for $|x| < 1$, $|y| < \infty$. Hence we have triplets of linearly independent solutions (u_0, v_1, w_1) (in the domain $|x| < |y|$ or $|x| < |x - y|$), (u_0, v_2, w_2) (in the domain $|y| < |x|$) and (u_0, v_3, w_3) (in the domain $|x - y| < |x|$).

On the other hand, in [4,5], we chose linearly independent solutions expressed as

(3) $$z_+ = (1 - e^{(\beta)})^{-1} \int_{C(x)} f(x, y, t)\,dt,$$

(4) $$z_0 = (1 - e^{(\gamma-\beta-\beta')})^{-1} \int_{C(0)} f(x, y, t)\,dt,$$

(5) $$z_- = (1 - e^{(\beta')})^{-1} \int_{C(y)} f(x, y, t)\,dt,$$

with

(6) $f(x, y, t) = t^{\beta+\beta'-\gamma}(t - x)^{-\beta}(t - y)^{-\beta'} e^t,$

and examined the asymptotic behaviour of them near the singular loci $x = \infty$, $y = \infty$ of irregular type. Here the paths of integration and the branch of the integrand are taken in such a way that, in the case where

(7) $$0 < \arg x < \pi < \arg y < 2\pi,$$
$$\pi < \arg(y - x) < 2\pi,$$

they have the following properties:

- (i) $C(a)$ $(a = 0, x, y)$ is a loop which starts from $t = -\infty$, encircles $t = a$ in the positive sense, and ends at $t = -\infty$.
- (ii) $C(x)$ lies over $C(0)$, and $C(y)$ lies under $C(0)$ in the t-plane.
- (iii) The branch of $f(x, y, t)$ is taken such that $\arg t = \arg(t - x) = \arg(t - y) = \pi$ at the end point $t = -\infty$ of each path of integration.

In this paper, we calculate connection

formulae for these solutions. Combining our result with [4,5], we can see the global behaviour of them in $\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$.

2. Result. Let $z = z(x, y)$ be a column vector function defined by ${}^t(z_+, z_0, z_-)$. Then we have the following result.

Theorem. *We have* $u_0 = \boldsymbol{a}_0 z$, $v_j = \boldsymbol{b}_j z$, $w_j = \boldsymbol{c}_j z$ $(j = 1,2,3)$, *where* $\boldsymbol{a}_0$, $\boldsymbol{b}_j$, $\boldsymbol{c}_j$ *are row vectors listed below*:

$$\boldsymbol{a}_0 = \frac{\Gamma(\gamma)}{2\pi i}(1 - e^{(\beta)}, e^{(\beta)} - e^{(\gamma-\beta')}, e^{(\gamma-\beta')} - e^{(\gamma)})$$
$$= -\Gamma(\gamma)\Big(\frac{e^{\beta\pi i}}{\Gamma(\beta)\Gamma(1-\beta)}, \frac{e^{(\gamma+\beta-\beta')\pi i}}{\Gamma(\gamma-\beta-\beta')\Gamma(1-\gamma+\beta+\beta')}, \frac{e^{(2\gamma-\beta')\pi i}}{\Gamma(\beta')\Gamma(1-\beta')}\Big),$$

$$\boldsymbol{b}_1 = \frac{e^{(\beta-\beta')\pi i}\Gamma(\beta'-\gamma+2)}{\Gamma(1-\beta)\Gamma(\beta+\beta'-\gamma+1)}(1, -1, 0),$$

$$\boldsymbol{b}_2 = \frac{e^{(\beta-\beta')\pi i}\Gamma(\beta-\gamma+2)}{\Gamma(1-\beta')\Gamma(\beta+\beta'-\gamma+1)}(0, 1, -1),$$

$$\boldsymbol{b}_3 = \frac{e^{-\beta'\pi i}\Gamma(2-\beta-\beta')}{\Gamma(1-\beta)\Gamma(1-\beta')}(1, 0, -1),$$

$$\boldsymbol{c}_1 = \frac{e^{-\gamma\pi i}\Gamma(\gamma-\beta')\Gamma(2-\gamma)}{2\pi i\Gamma(1-\beta')} \times (e^{(\beta)} - 1, e^{(\gamma-\beta')} - e^{(\beta)}, 1 - e^{(\gamma-\beta')})$$
$$= \frac{\Gamma(\gamma-\beta')\Gamma(2-\gamma)}{\Gamma(1-\beta')}\Big(\frac{e^{(\beta-\gamma)\pi i}}{\Gamma(\beta)\Gamma(1-\beta)}, \frac{e^{(\beta-\beta')\pi i}}{\Gamma(\gamma-\beta-\beta')\Gamma(1-\gamma+\beta+\beta')}, -\frac{e^{-\beta'\pi i}}{\Gamma(\gamma-\beta')\Gamma(1-\gamma+\beta')}\Big),$$

$$\boldsymbol{c}_2 = \frac{e^{-\gamma\pi i}\Gamma(\gamma-\beta)\Gamma(2-\gamma)}{2\pi i\Gamma(1-\beta)} \times (e^{(\beta)} - e^{(\gamma)}, e^{(\gamma-\beta')} - e^{(\beta)}, e^{(\gamma)} - e^{(\gamma-\beta')})$$
$$= \frac{\Gamma(\gamma-\beta)\Gamma(2-\gamma)}{\Gamma(1-\beta)}\Big(\frac{e^{\beta\pi i}}{\Gamma(\beta-\gamma)\Gamma(1-\beta+\gamma)}, \frac{e^{(\beta-\beta')\pi i}}{\Gamma(\gamma-\beta-\beta')\Gamma(1-\gamma+\beta+\beta')}, \frac{e^{(\gamma-\beta')\pi i}}{\Gamma(\beta')\Gamma(1-\beta')}\Big),$$

$$\boldsymbol{c}_3 = \frac{e^{-\gamma\pi i}\Gamma(\gamma-\beta-\beta')\Gamma(2-\gamma)}{2\pi i\Gamma(1-\beta-\beta')} \times (e^{(\beta)} - e^{(\gamma)}, e^{(\gamma-\beta')} - e^{(\beta)}, e^{(\gamma)} - e^{(\gamma-\beta')})$$
$$= \frac{\Gamma(\gamma-\beta-\beta')\Gamma(2-\gamma)}{\Gamma(1-\beta-\beta')} \times \Big(\frac{e^{\beta\pi i}}{\Gamma(\beta-\gamma)\Gamma(1-\beta+\gamma)}, \frac{e^{(\beta-\beta')\pi i}}{\Gamma(\gamma-\beta-\beta')\Gamma(1-\gamma+\beta+\beta')}, \frac{e^{(\gamma-\beta')\pi i}}{\Gamma(\beta')\Gamma(1-\beta')}\Big).$$

3. Proof of Theorem. For example, we verify the relation $w_1 = \boldsymbol{c}_1 z$. The others are shown by similar arguments. By the theorem of identity, it is sufficient to show the relation for (x, y) satisfying (7) and $|y| > |x|$. By [4; Corollary 2.3, (2) and §5.5], we have $z(x, ye^{2\pi i}) = M_2 M_0 z(x, y)$ in the domain $|y| > |x|$, where

(8) $M_2 M_0 =$
$$\begin{pmatrix} e^{(-\beta')} & 0 & 1 - e^{(-\beta')} \\ 0 & e^{(-\beta')} & 1 - e^{(-\beta')} \\ e^{(\beta-\gamma)} - e^{(-\gamma)} & e^{(-\beta')} - e^{(\beta-\gamma)} & 1 - e^{(-\beta')} + e^{(-\gamma)} \end{pmatrix}.$$

Since $w_1 = w_1(x, y)$ satisfies $w_1(x, ye^{2\pi i}) = e^{(-\gamma)} w_1(x, y)$, it follows that $\boldsymbol{c}_1 M_2 M_0 = e^{(-\gamma)} \boldsymbol{c}_1$. Hence $\boldsymbol{c}_1$ is written in the form

(9) $\boldsymbol{c}_1 = \kappa(e^{(\beta)} - 1, e^{(\gamma-\beta')} - e^{(\beta)}, 1 - e^{(\gamma-\beta')})$,

for some complex constant κ. To calculate κ, we may assume that $\operatorname{Re}\beta < 0$, $\operatorname{Re}\beta' < 0$, $\operatorname{Re}(\beta + \beta' - \gamma) > 0$. Substituting (3), (4), (5) and (9) into $w_1 = \boldsymbol{c}_1 z$, and putting $x = 0$, we have

(10) $y^{1-\gamma}(1 + O(y))$
$$= \kappa(e^{(\gamma-\beta')} - 1)\int_0^y t^{\beta'-\gamma}(t-y)^{-\beta'} e^t dt$$

near $y = 0$, where the path of integration is a segment from $t = 0$ to $t = y$, and the branch of the integrand is taken such that $\arg t = \arg y$, $\arg(t - y) = \arg y - \pi$ $(\pi < \arg y < 2\pi)$ along it. If we put $t = ys$ in (10), then $\arg s = 0$, $t - y = e^{-\pi i} y(1 - s)$, where $\arg(1 - s) = 0$ for $0 < s < 1$. Hence (10) is written in the form

$$\kappa e^{\beta'\pi i}(e^{(\gamma-\beta')} - 1)\int_0^1 s^{\beta'-\gamma}(1-s)^{-\beta'} e^{ys} ds = 1 + O(y),$$

from which we derive

$$\kappa = \frac{e^{-\gamma\pi i}\Gamma(\gamma-\beta')\Gamma(2-\gamma)}{2\pi i\Gamma(1-\beta')}.$$

Thus we have obtained the desired relation.

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