On Hasse Zeta Functions of Enveloping Algebras of Solvable Lie Algebras 2

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1. Introduction. 1.1. In our previous papers [3], [4], we have studied the Hasse zeta functions $\zeta_A(s)$ for finitely generated rings A over the ring $\boldsymbol{Z}$ of integers. These functions which have been known for commutative A, were generalized in [3] to the case where A is not necessarily commutative (concerning the definition, see 1.3 below). In [4], we studied them for the following case. Let R be a finitely generated commutative ring over $\boldsymbol{Z}$. Let $\mathfrak{g}$ be a solvable Lie algebra over R which is free of finite rank n as an R-module, and let A be the universal enveloping algebra of $\mathfrak{g}$ over R. In [4], we proved $\zeta_A(s) = \zeta_R(s-n)$ under certain conditions on p-mappings on $\mathfrak{g}$.

Now we shall show that these conditions can be eliminated, that is, we have

Theorem 1.2. *Let R, $\mathfrak{g}$, n, and A be as above. Then*

$$\zeta_A(s) = \zeta_R(s-n).$$

Theorem 1.2 follows from Theorem 1.4 below. Before stating it, we shall first review the definition of the function $\zeta_A(s)$.

1.3. For a (not necessarily commutative) finitely generated ring A over $\boldsymbol{Z}$, the Hasse zeta function $\zeta_A(s)$ of A is defined by

$$\zeta_A(s) = \prod_{r\geq 1} \zeta_{A,r}(s)$$

where r runs over integers ≥ 1 and,

$$\zeta_{A,r}(s) = \prod_{p} \exp \sum_{n=1}^{\infty} \frac{\# \mathfrak{S}_{A,r}(\boldsymbol{F}_{p^n})}{n} (p^{-s})^n$$

where $\mathfrak{S}_{A,r}$ is a certain scheme of finite type over $\boldsymbol{Z}$, p runs over prime numbers, and $\boldsymbol{F}_{p^n}$ is a finite field with p^n elements, so the function $\zeta_{A,r}(s)$ coincides with the product of Weil's zeta functions of $\mathfrak{S}_{A,r} \otimes_{\boldsymbol{Z}} \boldsymbol{F}_p$ [2] for all prime numbers p. For the algebraic closure K of $\boldsymbol{F}_p$, $\mathfrak{S}_{A,r}(K)$ is identified with the set of the isomorphism classes of all r-dimensional irreducible representations of A over K, and $\mathfrak{S}_{A,r}(\boldsymbol{F}_{p^n})$ is identified with the $\mathrm{Gal}(K/\boldsymbol{F}_{p^n})$-fixed part of $\mathfrak{S}_{A,r}(K)$.

We may assume that R is a finite field of characteristic $p>0$, for $\zeta_R(s)$, $\zeta_A(s)$ are products of $\zeta_{R/\mathfrak{m}}(s)$, $\zeta_{A/\mathfrak{m}A}(s)$ over all maximal ideals $\mathfrak{m}$ of R, respectively. So assume R is a finite field k of characteristic p.

Theorem 1.4. *Let k be a finite field. Let B be a finitely generated algebra over k. Let δ be a k-derivation of B, and let A be the ring $\{\sum_{i=0}^{N} b_i t^i \,;\, N \geq 0,\ b_i \in B\}$ in which t is an indeterminate and the multiplication is given by $tb - bt = \delta(b)\,(b \in B)$. Then*

$$\zeta_A(s) = \zeta_B(s-1).$$

As we may assume that the ring R is a finite field k, and as $\mathfrak{g}$ is a solvable Lie algebra, there exists a sequence of subalgebras of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_n = \{1\}$$

where $\mathfrak{g}_i$ is of dimension $n-i$ as a k-vector space of $\mathfrak{g}$, and $[\mathfrak{g}_{i-1}, \mathfrak{g}_i] \subset \mathfrak{g}_i$ for $1 \leq i \leq n$. Take the universal enveloping algebras of $\mathfrak{g}_{i-1}$ and $\mathfrak{g}_i$ as A and B, respectively, and apply Theorem 1.4 inductively, then we obtain Theorem 1.2. Hence it is sufficient to prove Theorem 1.4. But as the proof of Theorem 1.4 is complicated, we shall give here its proof only for the case that B is commutative, leaving the general proof for another publication.

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2. Proof of Theorem 1.4 (in the case that B is commutative). **2.1.** Let k, A, and B be as in the assumption of Theorem 1.4. As described before, in this section, let B be furthermore commutative. Let K be the algebraic closure of k.

Let $\mathfrak{S}_A = \coprod_{r\geq 1} \mathfrak{S}_{A,r}$, and for an extension k' of k, let $\mathfrak{S}_A^k(k')$ be the set of k'-rational points of $\mathfrak{S}_A$ as a k-scheme. Let $\mathfrak{S}_B = \mathrm{Spec}(B)$. We define $\mathfrak{S}_B^k(k')$ as we defined $\mathfrak{S}_A^k(k')$. It is sufficient to show that as a $\mathrm{Gal}(K/k)$-set (i.e. as a set endowed with an action of $\mathrm{Gal}(K/k)$),

$$\mathfrak{S}_A^k(K) \simeq \mathfrak{S}_B^k(K) \times K.$$

2.2. Let M be a finite dimensional irreducible representation of A over K. Since B is a commutative ring, there is a non-zero element v of M which is an eigen vector of all elements of B. We call such an element v a B-eigen vector. Let χ_v be the k-homomorphism from B into K which takes the eigen values of the elements of B concerning the eigen vector v. That is, $bv = \chi_v(b)v$ for any element b of B. We have the following lemma.

Lemma 2.2.1. *The k-homomorphism χ_v defined above does not depend on the choice of a B-eigen vector v.*

Proof. We show that for any $x \in M$, $b \in B$, and for any B-eigen vector v in M, there exists an integer $n \geq 1$ such that $(b - \chi_v(b))^n(x) = 0$. Let M' be the K-vector subspace of M defined by

$$M' = \{x \in M \,;\, \text{for any } b \in B, \text{ there is } n \geq 1 \text{ such that } (b - \chi_v(b))^n(x) = 0\}.$$

This M' is an A-submodule of M and is not 0. So $M' = M$. □

By Lemma 2.2.1, we write χ instead of χ_v.

Lemma 2.3.1. *There exists an integer $l \geq 0$ such that $\chi \circ \delta^{pi}$ $(0 \leq i \leq l-1)$ are linearly independent over K and $\chi \circ \delta^{pi}$ $(0 \leq i \leq l)$ are linearly dependent over K.*

Proof. Let k' be the image of $\chi : B \to K$. Since B is finitely generated algebra over k, k' is a finite field. Since δ^{p^i} is a derivation (see [1], Chapter 1, Proposition 1.3), the map $\chi \circ \delta^{p^i}$ from B to k' is determined by the values on the generators of B over k. Hence $\{\chi \circ \delta^{p^i} ; i \in \boldsymbol{Z}, i \geq 0\}$ is a finite set. This proves Lemma 2.3.1. □

Proposition 2.3.2. *Let l be as above. The dimension of M over K is p^l.*

To prove Proposition 2.3.2, it is sufficient to show Proposition 2.3.5 below. We prove some lemmas first.

Lemma 2.3.3. *Let $f : \{x \in \boldsymbol{Z} \,;\, x > 0\} \to \{x \in \boldsymbol{Z} \,;\, x > 0\}$ be a function defined by $f(x) = x - p^N$ where $p^N \,||\, x$. For an integer $m > 0$, and for any $b \in B$,*

$$(\chi(b) - b)t^m v = \alpha \chi \circ \delta^{m-f(m)}(b) t^{f(m)} v + (\text{a linear combination of the elements } t^i v \ (0 \leq i < f(m)))$$

where $\alpha \in \boldsymbol{F}_p$, $\alpha \neq 0$.

Proof. See [1] Chapter 1, Proposition 1.3. □

Lemma 2.3.4. *Assume that $\chi \circ \delta^{pi}$ $(0 \leq i \leq l-1)$ are linearly independent over K. Let $c_i \in K$ $(1 \leq i \leq p^l - 1)$, and assume that*

$$(b - \chi(b)) \left(\sum_{i=1}^{p^l-1} c_i t^i v\right) = 0 \text{ for any } b \in B.$$

Then $c_i = 0$ for $1 \leq i \leq p^l - 1$.

Proof. Let f be as in Lemma 2.3.3. We fix an integer $r \geq 0$. We prove $c_i = 0$ for all i such that $f(i) = r$, by downward induction on r. Assume that if $f(i) > r$, then $c_i = 0$. We prove that then $c_i = 0$ for i such that $f(i) = r$. We consider the coefficient of $t^r v$ in $(b - \chi(b))(\sum_{i=1}^{p^l-1} c_i t^i v)$. By Lemma 2.3.3 and by the assumption of the induction, it is the summation of the elements $\alpha_i c_i \chi \circ \delta^{i-r}(b)$ $(\alpha_i \in \boldsymbol{F}_p, \alpha_i \neq 0)$ over i such that $f(i) = r$. By assumption of this lemma, the coefficient of $t^r v$ in $(b - \chi(b))(\sum_{i=1}^{p^l-1} c_i t^i v)$ is 0, that is,

$$\sum_{i \in f^{-1}(r)} \alpha_i c_i \chi \circ \delta^{i-r}(b) = 0 \ (\alpha_i \in \boldsymbol{F}_p, \alpha_i \neq 0)$$

for any $b \in B$. By the definition of f, for i such that $f(i) = r$, $i - r = p^m$ for some $m \in \boldsymbol{Z}$, $0 \leq m \leq l-1$. Since $\chi \circ \delta^{pi}$ $(0 \leq i \leq l-1)$ are linearly independent over K, $c_i = 0$ for all i such that $f(i) = r$. Hence this proves Lemma 2.3.4. □

Proposition 2.3.5. (1) *Assume that $\chi \circ \delta^{pi}$ $(0 \leq i \leq l-1)$ are linearly independent over K. Then $v, tv, \ldots, t^{p^l-1}v$ are linearly independent over K.*

(2) *Assume that $\chi \circ \delta^{pi}$ $(0 \leq i \leq l \ )$ are linearly dependent over K. Then the dimension of M is $\leq p^l$.*

Proof. (1) By Lemma 2.3.4, the result follows.

(2) Assume that $\chi \circ \delta, \ldots, \chi \circ \delta^{p^l}$ are linearly dependent over K. We may assume that

$$\chi \circ \delta^{p^l} = \sum_{i=0}^{l-1} c_i \chi \circ \delta^{p^i} \ (c_i \in K). \quad (*)$$

Let $t' = t^{p^l} - \sum_{i=0}^{l-1} c_i t^{p^i}$. From the above equation $(*)$ and $t^{p^i}b - bt^{p^i} = \delta^{p^i}(b)$ for any $i \geq 0$ ([1], Chapter 1, Proposition 1.3), $(\chi(b) - b)t'v = 0$ for any $b \in B$. We consider the subspace $W = \{w \in M \,;\, (\chi(b) - b)w = 0, b \in B\}$ of M. This subspace is stable under the actions of elements of B and t'. So there is a B-eigen vector $v' \in W$ which is also an eigen vector of t'. Since the K-subspace of M which is spanned by $\{v', tv', \ldots, t^{p^l-1}v'\}$ is stable under the actions of the elements of B and t, it coincides with M. This proves (2). □

2.4. Let l be as in 2.3.

Lemma 2.4.1. *Let v be a B-eigen vector of*

M. Then any B-eigen vector of M has the form av where $a \in K$.

Proof. Any B-eigen vector w can be expressed as

$$w = cv + \sum_{i=1}^{p^l-1} c_i t^i v \quad \text{where}$$
$$c, c_i \in K \ (1 \le i \le p^l - 1).$$

We have $(\chi(b) - b)w = 0$ for all $b \in B$. By Lemma 2.3.4, $c_i = 0$ for $i = 1, \dots, p^l - 1$. This proves 2.4.1. □

From the above argument, we have that the irreducible representation M is determined by χ and $t^{p^l}v$. Write $\chi \circ \delta^{p^l} = \sum_{i=0}^{l-1} c_i \chi \circ \delta^{p^i}$ $(c_i \in K)$, and let $t' = t^{p^l} - \sum_{i=0}^{l-1} c_i t^{p^i}$. Then by the proof of Proposition 2.3.5 (2) and by Lemma 2.4.1, v is an eigen vector of t'. Hence $t^{p^l}v = cv + \sum_{i=0}^{l-1} c_i t^{p^i} v$ $(c, c_i \in K)$. We can take c arbitrarily. So we see that as a $\mathrm{Gal}(K/k)$-set, $\mathfrak{S}_A^k(K) \simeq \mathfrak{S}_B^k(K) \times K$; the class of $M \mapsto (\chi, c)$. This proves Theorem 1.4.

References

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