Inverse Mapping Theorem in the Ultradifferentiable Class

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(Communicated by Kiyosi ITÔ, M.J.A., Oct. 14, 1996)

The main purpose of this paper is to give a simple proof of a result similar to the inverse mapping theorem of Komatsu [1] under a weaker condition than that of [1], including the infinite dimensional case treated in Yamanaka [3].

In [1],[3] the majorant series method and the Lagrange formula are used, and [3] uses a generalization of the higher order chain rule of Faa'di Bruno. Here neither the majorant series method nor the higher order chain rule is utilized. Alternatively we prove and use a generalization of a result in Rudin [2] and a variant of the Lagrange formula (Theorem 3 below).

Let M_p, $p = 0,1,2,\cdots$, be a sequence of positive numbers with $M_1 = 1$. Let X, Y be Banach spaces and U an open subset of X. A map $f: U \to Y$ is said to belong to the ultradifferentiable class $\{M_p\}$ (or $\{M_p\}(U, Y)$), if $f \in C^\infty(U, Y)$ in the sense of Fréchet-differentiation and if there are constants C and h such that

$$\| f^{(p)}(x) \| \leq Ch^p M_p, \quad p = 0,1,2,\cdots, x \in U.$$

In [2], [3] the following condition is considered: There is a constant H such that

$$\text{(1)} \qquad N_p^{1/p} \leq HN_q^{1/q} \quad \text{if } 1 \leq p \leq q,$$

where

$$N_p = \frac{M_p}{p!}.$$

Here we consider the condition that there is a constant H such that the inequality

$$\text{(2)} \qquad \prod_{i=1}^{j} N_{k_i} \leq H^n N_n$$

holds for positive integers k_i with $\sum_{i=1}^{j} k_i = n$, $n = 1,2,\cdots$, $j = 1,2,\cdots, n$.

This condition follows from (1).

Example. For $n = 1,2,\cdots$, let

$$M_n = \begin{cases} n!n^{n(n-1)} & (n = 2^m, m = 0,1,\cdots) \\ n!n^{n(n+1)} & \text{(otherwise)}. \end{cases}$$

Then this sequence $\{M_p\}$ satisfies (2) with $H = 1$ but not the condition (1). In fact we have $\sup\{N_{n-1}^{1/(n-1)}/N_n^{1/n}; n = 2^m, m \geq 1\} = \infty$. On the other hand, if $\sum_{i=1}^{j} k_i = n$ and $1 \leq k_i < n - 1$, then

$$\prod_{i=1}^{j} N_{k_i} \leq \prod_{i=1}^{j} k_i^{k_i(k_i+1)} \leq \prod_{i=1}^{j} n^{k_i(n-1)} \leq N_n.$$

If $k_r = n - 1$ for some r, then $j = 2$ and $k_s = 1$ $(s \neq r)$, hence $\prod_{i=1}^{j} N_{k_i} = N_{n-1} \leq N_n$.

Thus (2) is strictly weaker than (1).

It is shown in [2] that the class $\{M_p\}$ is closed under division (in the one-dimensional case) if M_p satisfies (1). Here we have the following generalization of this.

Theorem 1. Assume (2). Let X, Y and Z be Banach spaces and U an open subset of X. If T belongs to the class $\{M_p\}(U, L(Z, Y))$ and $T(a): Z \to Y$ is bijective for a point a in U, then the map $x \mapsto [T(x)]^{-1}$ belongs to the class $\{M_p\}(U_0, L(Y, Z))$ for some open subset U_0 of U containing a.

Proof. By assumption we have

$$\| T^{(k)}(x) \| \leq h^{k+1} M_k, \quad k = 0,1,2,\cdots,$$

with some constant h. The open mapping theorem implies that $[T(a)]^{-1}$ belongs to $L(Y, Z)$. There exists an open set U_0 containing a such that, for $x \in U_0$, $[T(x)]^{-1}$ coincides with

$$R(x) = [T(a)]^{-1} \sum_{j=0}^{\infty} \{(T(a) - T(x))[T(a)]^{-1}\}^j,$$

which belongs to $L(Y, Z)$ and $\| R(x) \| \leq C$ for a constant C. By the boundedness of derivatives of T and by the Leibniz rule, the series

$$R(u) = R(x) \sum_{j=0}^{\infty} [(T(x) - T(u))R(x)]^j$$

may be differentiated with respect to u in a neighborhood of x, term by term any number of times, since the resulting series converge uniformly in the neighborhood of x. Putting $u = x$ after differentiating this equality n-times by u, we have

$$R^{(n)}(x) = R(x) \sum_{j=1}^{n} \sum n! \prod_{i=1}^{j} \frac{1}{k_i!} [- T^{(k_i)}(x)R(x)],$$

where $\sum$ denotes the summation with respect to positive integers k_i with $\sum_{i=1}^{j} k_i = n$. Thus (2) implies

$$\| R^{(n)}(x) \| \leq C \sum_{j=1}^{n} \sum n! \prod_{i=1}^{j} Ch^{k_i+1} \frac{M_{k_i}}{k_i!}$$

$$\leq \sum_{j=1}^{n} \sum C^{j+1} h^{n+j} H^n n! N_n,$$

hence

$$\| R^{(n)}(x) \| \leq \sum_{j=1}^{n} \binom{n-1}{j-1} C^{j+1} h^{n+j} H^n M_n$$
$$\leq C^2 H^n h^{n+1} (Ch+1)^{n-1} M_n.$$

Therefore R belongs to $\{M_p\}(U_0, L(Y, Z))$.

Now we assume that there is a constant H such that

(3) $\prod_{i=1}^{j} N_{k_i+1} \leq H^n N_{n+1}$ if $\sum_{i=1}^{j} k_i = n$, $k_i \geq 0$,
$n = 1,2,\cdots, j = 1,2,\cdots, n.$

This condition is equivalent to the condition that M_{p+1} satisfies (2) since $1 \leq \Pi_{i=1}^{j}(k_i+1) \leq 2^n$. (3) is strictly weaker than the condition

(4) $N_p^{1/(p-1)} \leq H N_q^{1/(q-1)}$ if $2 \leq p \leq q$,

which is assumed in [1].

We have the following inverse mapping theorem.

Theorem 2. Assume (3). Let X, Y be Banach spaces and U an open subset of X. Let f belong to $\{M_p\}(U, Y)$ and $f'(a) : X \to Y$ be bijective for a point $a \in U$. Then there exist open sets $U_0 \subset U$, $V_0 \subset Y$ such that $a \in U_0$, $f(a) \in V_0$ and $f : U_0 \to V_0$ is a C^∞-diffeomorphism and the inverse map f^{-1} of f belongs to $\{M_p\}(V_0, X)$.

We know by the well-known inverse mapping theorem for C^∞-maps that there exist open sets $U_0 \subset U$ and $V_0 \subset Y$ such that $a \in U_0$ and $f : U_0 \to V_0$ is a C^∞-diffeomorphism. Therefore it only remains to estimate the derivatives of the inverse map f^{-1} of f. In order to estimate them we use a variant of the Lagrange formula:

Let R_j, $j = 1,2,\cdots$, be in $C^\infty(U, L(Y, X))$, where U is an open subset of X. We define S_n, $n = 0,1,\cdots$, recursively by

$$S_0(x) = I_X : X \to X \text{ (identity) and}$$
$$S_n(x) = (S_{n-1}(x) R_n(x))' \ (n \geq 1)$$

for $x \in U$. Here S_{n-1} belongs to $C^\infty(U, L(X, L^{n-1}(Y, X)))$ and accordingly $S_{n-1}(x)R_n(x) \in L^n(Y, X)$, where $L^n(Y, X)$ denotes the set of all bounded multi-n-linear maps from Y^n to X.

We can easily see that

$$\| S_n(x) \| \leq \sum^{(n)} A(k_1, \cdots, k_n) \prod_{i=1}^{n} \| R_i^{(k_i)}(x) \|,$$

where $\sum^{(n)}$ denotes the summation with respect to nonnegative integers k_i with $\sum_{i=1}^{n} k_i = n$ and $A(k_1, \cdots, k_n)$ are nonnegative integers with

$$\sum^{(n)} A(k_1, \cdots, k_n) \prod_{i=1}^{n} t_i^{k_i} = \prod_{j=1}^{n} \Big(\sum_{i=1}^{j} t_i \Big),$$

where $t_1, \cdots, t_n$ are independent (real-valued) variables. Comparing the right-side of the last equality with the polynomial

$$\Big(\sum_{i=1}^{n} t_i \Big)^n = \sum^{(n)} n! \prod_{i=1}^{n} \frac{1}{k_i!} t_i^{k_i},$$

we get $A(k_1, \cdots, k_n) \leq n!/(k_1! \cdots k_n!)$ and accordingly

$$\| S_n(x) \| \leq \sum^{(n)} n! \prod_{i=1}^{n} \frac{\| R_i^{(k_i)}(x) \|}{k_i!}. \tag{5}$$

If $R_j = R$ for all j, we write $S_n[R](x) = S_n(x)$.

Theorem 3. Let f be a map as in Theorem 2. Put $R(x) = [f'(x)]^{-1} \in L(Y, X)$ and $g = f^{-1}$. Then

(6) $g^{(n)}(y) = S_{n-1}[R](x) R(x)$, $n \geq 1$
$(x = g(y))$.

Proof. Since $g'(y) = [f'(x)]^{-1} = R(x)$, (6) holds for $n = 1$. For every smooth map v, we have

$$\frac{dv}{dy} = \frac{dv}{dx} R(x).$$

Thus (6) is verified immediately by induction on n.

Proof of Theorem 2. Applying Theorem 1 to $T(x) = f'(x)$, we obtain that $R \in \{M_{p+1}\}(U_0, L(Y, X))$, and thus $\| R^{(k)}(x) \| \leq Ch^k M_{k+1}$ with some constants C and $h (k = 0,1,\cdots)$. (6) and (5) yield

$$\| g^{(n+1)}(y) \| \leq C \sum^{(n)} n! \prod_{i=1}^{n} \frac{\| R^{(k_i)}(x) \|}{k_i!}$$
$$\leq C^{n+1} h^n n! \sum^{(n)} \prod_{i=1}^{n} \frac{M_{k_i+1}}{k_i!},$$

and hence (3) implies

$$\| g^{(n+1)}(y) \| \leq C^{n+1} h^n n! H^n N_{n+1} \sum^{(n)} \prod_{i=1}^{n} (k_i + 1).$$

Since

$$\sum^{(n)} \prod_{i=1}^{n} (k_i+1) \leq 2^n \binom{2n-1}{n} \leq 2^{3n},$$

we have $\| g^{(n+1)}(y) \| \leq C(8ChH)^n M_{n+1}$, therefore g belongs to the class $\{M_p\}(V_0, X)$.

References

[1] H. Komatsu: The implicit function theorem for ultradifferentiable mappings. Proc. Japan Acad., **55A**, 69–72 (1979).

[2] W. Rudin: Division in algebras of infinitely differentiable functions. J. Math. Mech., **11**, 797–809 (1962).

[3] T. Yamanaka: Inverse map theorem in the ultra-F-differentiable class. Proc. Japan Acad., **65A**, 199–202 (1989).