Oscillation of Solutions of Nonlinear Wave Equations*)

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1. Introduction. In this paper we shall consider the sign of solutions of certain nonlinear wave equations subject to suitable homogeneous boundary conditions. It is related to the oscillation behavior of continuous finite bodies with respect to the time variable.

Let Ω be a bounded simply connected domain in $\boldsymbol{R}^n$ and $\partial\Omega$ be its smooth boundary.

We suppose all functions and solutions appeared in this paper to be real-valued. We denote $\dfrac{\partial}{\partial x_k}$ $(k = 1,2\ldots n)$ by ∂_k and $\dfrac{\partial}{\partial t}$ by ∂_t.

We shall consider the nonlinear wave equation

(1) $\mathscr{D}u = \partial_t(\alpha(t)\partial_t u) + \beta(t)\partial_t u + \mathscr{N}u = 0$ in $\Omega \times \boldsymbol{R}^+$,

and the homogeneous boundary condition

(2) $\mathscr{B}u(x, t) = 0$ on $\partial\Omega \times \boldsymbol{R}^+$,

where $\mathscr{N}$ is a nonlinear differential operator on x defined exactly afterwards.

When $\mathscr{N}$ is a linear elliptic differential operator on $x \in \Omega$, e.g. $-\Delta$ or Δ^2, the oscillating behavior is well investigated within the framework of the eigenvalue problems. For the linear case we refer to Chapter 5 and 6 of [4]. When $\mathscr{N}$ is nonlinear, it seems that the results have been obtained less compared with the linear case. Cazenave and Haraux have obtained some remarkable results (see [3] and [7]) when $\mathscr{N}$ is semilinear. In [12] results for simpler equations than those of this paper are stated. Besides them we refer to [2] and [9].

In this paper $\mathscr{N}$ is supposed to be more general than that of Cazenave and Haraux. We shall show that there exist different points (x_1, t_1) and (x_2, t_2) in $\Omega \times \boldsymbol{R}^+$ such as $u(x_1, t_1)u(x_2, t_2) < 0$, as is the unsatisfactory result for showing oscillation of u.

Elliptic differential operator of second order is typical of $\mathscr{N}$ and in this case we prescribe the boundary condition to be the homogeneous Dirichlet boundary condition, i.e. $u = 0$ on $\partial\Omega \times \boldsymbol{R}^+$. Besides we can consider $\mathscr{N}$ to be 2^m th order for $m = 2, \ldots$ with suitable boundary conditions. For simplicity we shall treat only the $m = 2$ case. Then we prescribe its boundary condition to be one concerned with a supported edge. Here we shall state the second order case in detail.

We don't prove the existence of solutions of initial-boundary value problems satisfying (1), (2) and suitable initial conditions with suitable compatibility conditions, but we suppose the existence of unique global solutions in time (see **A.2** in §3 and **A.5** in §4).

2. Preliminary results. In this section we shall prepare and collect several auxiliary results.

Let $\alpha, \beta, \gamma : \boldsymbol{R} \to \boldsymbol{R}$ be continuous, and α be a positive function of C^1. We define the ordinary differential operator l by

(3) $l(ly)(t) = (\alpha(t)y'(t))' + \beta(t)y'(t) + \gamma(t)y(t)$,

where $'$ means $\dfrac{d}{dt}$.

Lemma 2.1. *Let $x(t)$ and $y(t)$ satisfy $(lx)(t) \leq 0$ and $(ly)(t) = 0$ in $[t_0, \infty)$ associated with $x(t_0) = y(t_0)$ and $x'(t_0) = y'(t_0)$ for any fixed t_0, respectively. If $y(t) \geq 0$ and $x(t) \neq 0$ for $t \geq t_0$, then $x(t) \leq y(t)$ for $t \geq t_0$.*

Proof. Since

$y(lx) - x(ly) = \{\alpha(x'y - xy')\} + \beta(x'y - xy') \leq 0$,

we get

$$\alpha(t)(x'y - xy')(t)\exp\left(\int_{t_0}^{t}\frac{\beta(s)}{\alpha(s)}ds\right) \leq \alpha(t_0)(x'y - xy')(t_0) = 0,$$

whence $(x'y - xy')(t) \leq 0$. It follows from $(x'y - xy')(t) \leq 0$ and $x(t) \neq 0$ that

$$\left(\frac{y(t)}{x(t)}\right)' \geq 0.$$

Hence we have $x(t) \leq y(t)$ for $t \geq t_0$. Q.E.D.

In subsequent sections we shall apply the result which assures the existence of zeros of solutions of the differential equation $ly = 0$ to obtain

*) Dedicated to Prof. Yasushi Shizuta on his 60th birthday.

our theorems. There are many established results concerning the existence of zeros (see [6] and [8]). We adopt the following result.

Lemma 2.2. (*Leigton and Kreith*). *Let $\alpha \in C^1$, β, $\gamma \in C$ and $\alpha > 0$. If for any real number h,*

$$\int_h^\infty \frac{1}{\alpha(t)}\,dt = \infty \text{ and}$$

$$\lim_{t\to\infty}\left\{\frac{\beta(t)}{2\alpha(t)} + \int_h^t \left[\gamma(s) - \frac{\beta(s)^2}{4\alpha(s)}\right]ds\right\} = \infty.$$

Then every nontrivial solution of $ly = 0$ has an infinite number of zeros in every interval of the form $[h, \infty)$.

For the proof we refer to [8].

We state one more established result concerning the eigenvalue problem for second order elliptic partial differential operators. We give the selfadjoint differential operator $\mathscr{L}$ by

$$\mathscr{L}u = \sum_{i,j=1}^n \partial_i(a_{i,j}(x)\partial_j u + b_i(x)u) - \sum_{i=1}^n b_i(x)\partial_i u + c(x)u, \tag{4}$$

where $\mathscr{L}$ satisfies the following conditions

1. $a_{ij}(x) = a_{ji}(x)$ and $\mathscr{L}$ is selfadjoint in $H_0^1(\Omega)$,
2. there exists a positive number c such that $\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \geq c\,|\,\xi\,|^2$ for any $x \in \Omega$ and $\xi \in \boldsymbol{R}^n$,
3. $\mathscr{L}$ has bounded coefficients.

Lemma 2.3. *Let $\mathscr{L}$ be the elliptic operator defined in* (4). *Then $\mathscr{L}$ has a countably infinite discrete set of eigenvalues. The minimum eigenvalue λ is simple and has a positive smooth eigenfunction ϕ.*

For the detailed statement and the proof we refer to [5] or [7].

3. Second order case. In this section we shall treat the nonlinear second order wave equation.

Let

$$\mathscr{N}u = -\gamma(t)\mathscr{L}\{A(x, t\,; u)u\} + b(x, t\,; u)u, \tag{5}$$

where $\mathscr{L}$ is the operator defined in (4). We suppose that $A(x, t\,; u)$ and $b(x, t\,; u)$ are functions of x, t and some quantities related to u, i.e. u itself, its derivatives or/and its integrals etc. We shall state afterward the precise assumptions on $A(x, t\,; u)$ and $b(x, t\,; u)$.

Now we consider

$$\mathscr{D}u = \partial_t(\alpha(t)\partial_t u) + \beta(t)\partial_t u + \mathscr{N}u = 0 \text{ in } \Omega \times \boldsymbol{R}^+. \tag{6}$$

As a typical example of (6) we give the Kirchhoff equation

$$\partial_t^2 u - const.(1 + \|\nabla u\|^2)\Delta u = 0,$$

where $\|\nabla u\| = \int_\Omega \sum_{k=1}^n |\,\partial_k u(t, x)\,|^2 dx$. As a typical b we give $b(x, t\,; u) = u^{2p}$, where p is a natural number.

We shall investigate oscillating behavior of the solutions of Eq. (6) satisfying the homogeneous Dirichlet boundary condition

$$u(x, t) = 0 \text{ at } (x, t) \in \partial\Omega \times \boldsymbol{R}^+. \tag{7}$$

We set assumptions **A. 1, A. 2** and **A. 3**.

A.1 *The coefficients of $\mathscr{D}$, i.e. α, β, $\gamma : \boldsymbol{R} \to \boldsymbol{R}$ satisfy the conditions stated in* Lemma 2. 2.

A.2 *Let t_0 be any fixed number of $\boldsymbol{R}^+$ and (u_0, u_1) be any element of $V \times L^2(\Omega)$, where $V = H_0^1(\Omega)$. We consider the initial-boundary value problem* (6) *and* (7) *associated with the initial condition $u(x, t_0) = u_0(x)$ and $\partial_t u(x, t_0) = u_1(x)$. Then*

1. *there exists a global smoooth solution u such that*
$$u \in C(\boldsymbol{R}^+, V) \cap C^1(\boldsymbol{R}^+, L^2(\Omega)) \cap C^2(\boldsymbol{R}^+, V'),$$
where V' is the dual space of V,
2. *the uniqueness of the solution holds in the sense that if $u(x, t_0) = \partial_t u(x, t_0) = 0$, then $u(x, t)$ vanishes in $\Omega \times \boldsymbol{R}^+$.*

Concernig the results of the unique existence of global solutions of the initial-boundary value problems for nonlinear wave equations we refer to [7], [10], [11] and the references of [7].

A.3 1. *The minimum eigenvalue λ of $\mathscr{L}$ stated in* Lemma 2.3 *is not zero.*
2. *There exists a constant a such that $0 < a \leq A(x, t\,; u)$ if $\lambda > 0$ or $0 < A(x, t\,; u) \leq a$ if $\lambda < 0$.*
3. *$b(x, t\,; u) \geq 0$ for every (x, t, u).*

Now we state our theorem.

Theorem 3.1. *Suppose that* **A.1**, **A.2** *and* **A.3**. *Let u be the solution stated in* **A.2**. *If u does not vanish identically, there exist some (x_1, t_1) and $(x_2, t_2) \in \Omega \times \boldsymbol{R}^+$ such that $u(x_1, t_1)u(x_2, t_2) < 0$.*

Proof. *Let λ be the minimum eigenvalue of $\mathscr{L}$ and $\phi(x)$ be its corresponding positive eigenfunction.* We put

$$U(t) = \int_\Omega u(x, t)\phi(x)dx.$$

Then owing to 1. of **A.2** we have

$$U'(t) = \int_\Omega \partial_t u(x, t)\phi(x)dx \quad \text{and} \tag{8}$$

$$U''(t) = \int_\Omega \partial_t^2 u(x, t)\phi(x)dx.$$

By integrating by parts we have

$$\int_\Omega \mathscr{L}\{A(x, t; u)u\}\phi(x)dx = \int_\Omega A(x, t; u)u\mathscr{L}\phi(x)dx$$

Because of $\mathscr{L}\phi = -\lambda\phi$, we have

$$\int_\Omega \mathscr{L}\{A(x, t; u)u\}\phi(x)dx = -\lambda\int_\Omega A(x, t; u)u\phi(x)dx. \tag{9}$$

We multiply (6) by ϕ. Then by (8) and (9) we have

$$(\alpha(t)U')' + \beta(t)U' = -\lambda\gamma(t)\times \int_\Omega A(x, t; u)u\phi dx - \int_\Omega b(x, t; u)u\phi dx. \tag{10}$$

Let $u(x, t_0) > 0$ for any $x \in \Omega$ and $t_0 \in \boldsymbol{R}^+$. Then there exists some time interval I with t_0 as its left end point such that $u(x, t_0) > 0$ for any $(x, t) \in \Omega \times I$. We shall show the length of I to be finite.

When we suppose that $u(x, t_0) < 0$, we also get the same conclusion.

From $u(x, t_0) > 0$ in $\Omega \times I$ and $\phi(x) > 0$ in Ω we have $U(t) > 0$. From **A.3** we obtain in I

the right hand side of (10) $\leq -\lambda a\gamma(t)U(t)$

for either $\lambda > 0$ or $\lambda < 0$.

Thus we have the differential inequality in I

$$(\alpha(t)U'(t))' + \beta(t)U'(t) + a\lambda\gamma(t)U(t) \leq 0. \tag{11}$$

We shall show that there exists some finite $T > t_0$ such that $U(T) = 0$.

We consider the ordinary differential equation for $t \geq t_0$

$$\begin{cases} (\alpha(t)v'(t))' + \beta(t)v'(t) + a\lambda\gamma(t)v(t) = 0 \\ v(t_0) = U(t_0) \text{ and } v'(t_0) = U'(t_0). \end{cases} \tag{12}$$

Since α, β and $a\lambda\gamma$ satisfy the conditions of Lemma 2. 1, we get

$$0 < U(t) \leq v(t) \text{ in } I.$$

If I is an infinite interval, there exists $t_1 \in I$ such that $v(t_1) = 0$ because α, β and $a\lambda\gamma$ satisfy the conditions of Lemma 2.2. Hence there exists $t_2 (\leq t_1)$ in I such that $U(t_2) = 0$. It is contradiction. Thus I is finite, whence there exists $T (> t_0)$ such that $U(T) = 0$.

Let $I_\varepsilon = (T - \varepsilon, T + \varepsilon)$ be an interval in $\boldsymbol{R}^+$ for any positive ε. If $u(x, t) < 0$ for some $(x, t) \in \Omega \times I_\varepsilon$, we have the desired result. We suppose that $u \geq 0$ almost everywhere in $\Omega \times I_\varepsilon$. After this we omit "almost everywhere" in this paragraph. From $U(T) = 0$ we get $u(\cdot\ , T) = 0$ in Ω. Then we can say that $\partial_t u(\cdot\ , T) = 0$ holds in Ω. We shall show the fact. Because of $u \in C^1(\boldsymbol{R}^+, L^2)$ there exists $\partial_t u(\cdot, T)$ in $L^2(\Omega)$ and a suitable sequence $\{h_n\}$ with $|h_n| < \varepsilon$ such that

$$\lim_{h_n \to 0} \frac{u(x, T + h_n) - u(x, T)}{h_n} - \partial_t u(x, T) = 0 \quad \text{as } h_n \to 0. \tag{13}$$

Nothing that $u(x, T) = 0$ and $u(x, t) \geq 0$ in $\Omega \times I_\varepsilon$, it follows from (13) that $\partial_t u(x, T) \leq 0$ if $h_n > 0$ and $\partial_t u(x, T) \geq 0$ if $h_n < 0$. Hence we get $\partial_t u(x, T) = 0$ in Ω. Thus we have

$$u(\cdot, T) = \partial_t u(\cdot, T) = 0 \text{ in } \Omega.$$

Therefore it follows from the uniqueness of **A.2** that $u \equiv 0$ in $\Omega \times \boldsymbol{R}^+$.

Thus we have proved this theorem. Q.E.D.

4. Higher order case. In this section we let nonlinear operator $\mathscr{N}$ be higher even order than 2. If we suppose $\mathscr{N}$ and the boundary operator $\mathscr{B}$ satisfy suitable conditions, we can show the same result as in Theorem 3.1. For simplicity we shall consider our problem for the following equation.

$$\mathscr{N}u = \gamma(t)A_0(t; u)\Delta^2 u - \delta(t)\Delta\{A(x, t; u)u\} + b(x, t; u)u, \tag{14}$$

where Δ is the Lapalace operator. We assume that $A_0(t; u)$, $A(x, t; u)$ and $b(x, t; u)$ satisfy the following assumption.

A.4 1. $A_0(t; u)$ *is independent of* x *and continuous on* t.

2. *There exists a constant* a_0 *such that* $0 < a_0 \leq A_0(t; u)$.

3. $A(x, t; u)$ *and* $b(x, t; u)$ *satisfy the same conditions as those of* A.3.

We shall treat the following problem associated with a suitable initial condition.

$$\begin{cases} \mathscr{D}u = \partial_t(\alpha(t)\partial_t u) + \beta(t)\partial_t u + \mathscr{N}u = 0 \\ \qquad\qquad\qquad\qquad \text{in } \Omega \times \boldsymbol{R}^+ \\ u(x, t) = \Delta u(t, x) = 0 \text{ on } \partial\Omega \times \boldsymbol{R}^+. \end{cases} \tag{15}$$

In addition we set assumptions **A.5** and **A.6**

A.5 *For any smooth initial data* (u_0, u_1) *there exists a unique smooth solution* u *satisfying* (15).

A.6 λ *is the minimum eigenvalue stated in* **Lemma 2.3**. $\alpha(t)$, $\beta(t)$ *and* $a_0\lambda^2\gamma(t) + a\lambda\delta(t)$ *satisfy the same conditions as* $\alpha(t)$, $\beta(t)$ *and* $\gamma(t)$ *in* Lemma 2.2 *respectively.*

For the existence and the uniqueness of solutions concerning **A.5** we refer to [1].

Theorem 4.1. *Suppose that* **A.3**, **A.4**, **A.5** and **A.6**. *Let u be a smooth solution of* (15). *If u does not vanish identically, there exist some* (x_1, t_1) *and* $(x_2, t_2) \in \Omega \times \boldsymbol{R}^+$ *such that* $u(x_1, t_1)u(x_2, t_2) < 0$.

Proof. We can prove this theorem in a similar fashion as the argument in Theorem 3.1. We use the same eigenfunction ϕ and the corresponding eigenvalue $\lambda > 0$ as in Theorem 3.1 for $\mathscr{L} = \Delta$. We put

$$U(t) = \int_\Omega u(x, t)\phi(x)\,dx.$$

We multiply the equation $\mathscr{D}u = 0$ in (15) by ϕ and integrate it by parts. Then we have

$$(\alpha(t)U')' + \beta(t)U'$$
$$= -\lambda^2\gamma(t)\int_\Omega A_0(t\,; u)u\phi dx - \lambda\delta(t) \times$$
$$\int_\Omega A(x, t\,; u)u\phi dx - \int_\Omega b(x, t\,; u)u\phi dx.$$

Then letting $u(x, t_0) > 0$ in Ω for any fixed t_0, we have from **A.4**

$$(\alpha(t)U')' + \beta(t)U' + (a_0\lambda^2\gamma(t) + a\lambda\delta(t))U \leq 0.$$

By applying **A.5** and **A.6** instead of **A.1** and **A.2**, the rest of the argument is the exactly same as the corresponding part in the proof of Theorem 3.1 and will be omitted. Thus we have the desired result. Q.E.D.

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