The Automorphism Group of the Klein Curve in the Mapping Class Group of Genus 3

By Atsushi MATSUURA

Department of Mathematical Sciences, University of Tokyo

(Communicated by Heisuke HIRONAKA, M. J. A., Sept. 12, 1996)

Let R be a compact Riemann surface of genus $g \geq 2$. Then $\mathrm{Aut}(R)$, the automorphism group of R, can be embedded into the mapping class group (for its definition, see[**1**,Ch. 4]) or the Teichmüller group Γ_g of genus g;

(1) $\iota : \mathrm{Aut}(R) \hookrightarrow \Gamma_g \simeq \mathrm{Out}^+(\pi_1(R)) = \mathrm{Aut}^+(\pi_1(R))/\mathrm{Int}(\pi_1(R)).$

Here, $\mathrm{Aut}^+(\pi_1(R))$ consists of the automorphisms of $\pi_1(R)$ inducing the trivial action on $H_2(\pi_1(R), \boldsymbol{Z}) \simeq \boldsymbol{Z}$.

Recall the Hurwitz theorem, which states that

(2) $\# \mathrm{Aut}(R) \leq 84(g-1).$

If the equality holds in (2), then R is called a Hurwitz Riemann surface and $\mathrm{Aut}(R)$ is called a Hurwitz group.

Let X be the Klein curve of genus 3 defined by the equation

$$x^3y + y^3z + z^3x = 0.$$

It is well known that X is a Hurwitz Riemann surface; $G := \mathrm{Aut}(X)$ is isomorphic to $PSL_2(\boldsymbol{F}_7)$ and has order 168.

Now let us forget about the Klein curve, and consider an orientable compact C^∞ surface X of genus 3. We define the canonical generators of $\pi_1(X, b)$ with base point b as in the figure 1. They satisfy the fundamental relation

(3) $(\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1})(\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1})(\beta_3\alpha_3\beta_3^{-1}\alpha_3^{-1}) = 1.$

Let $\tilde{\varphi}_2$, $\tilde{\varphi}_3$, $\tilde{\varphi}_7$ be the elements of $\mathrm{Aut}^+(\pi_1(X))$ defined by

$$\tilde{\varphi}_2(\alpha_1) = \alpha_2\beta_2^{-1}\alpha_2^{-1}\alpha_1^{-1}\beta_3^{-1}\beta_2 \qquad \tilde{\varphi}_2(\beta_1) = \beta_2^{-1}\beta_3\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}$$
$$\tilde{\varphi}_2(\alpha_2) = \beta_3^{-1}\alpha_2^{-1} \qquad \tilde{\varphi}_2(\beta_2) = \alpha_2\beta_3\beta_2^{-1}\alpha_2^{-1}$$
$$\tilde{\varphi}_2(\alpha_3) = \alpha_2\beta_2^{-1}\alpha_2^{-1}\beta_1^{-1}\alpha_1^{-1}\alpha_3\alpha_2^{-1} \qquad \tilde{\varphi}_2(\beta_3) = \alpha_2\beta_3\alpha_2^{-1},$$

$$\tilde{\varphi}_3(\alpha_1) = \alpha_2\beta_3\alpha_3^{-1}\alpha_1\alpha_2\beta_2\alpha_2^{-1} \qquad \tilde{\varphi}_3(\beta_1) = \alpha_2\beta_2^{-1}\alpha_2^{-1}\alpha_1^{-1}\alpha_3\alpha_1\alpha_2\beta_2\alpha_2^{-1}$$
$$\tilde{\varphi}_3(\alpha_2) = \alpha_3^{-1}\alpha_1\beta_1\alpha_1^{-1} \qquad \tilde{\varphi}_3(\beta_2) = \alpha_1\beta_1^{-1}\alpha_1^{-1}\alpha_3\alpha_2\beta_2^{-1}\alpha_2^{-1}\beta_1\alpha_1^{-1}$$
$$\tilde{\varphi}_3(\alpha_3) = \alpha_2\beta_2\alpha_2\beta_2^{-1}\alpha_2^{-1}\beta_1 \qquad \tilde{\varphi}_3(\beta_3) = \alpha_1\beta_1^{-1}\alpha_1^{-1}\alpha_3\alpha_2\beta_2^{-1}\alpha_2^{-1}\beta_1,$$

$$\tilde{\varphi}_7(\alpha_1) = \beta_1^{-1}\alpha_1^{-1}\alpha_3\beta_3^{-1}\alpha_2^{-1} \qquad \tilde{\varphi}_7(\beta_1) = \alpha_2\beta_3\alpha_3^{-1}\alpha_1\alpha_2\beta_2\alpha_2^{-1}\beta_3^{-1}\alpha_2^{-1}$$
$$\tilde{\varphi}_7(\alpha_2) = \alpha_2\beta_2^{-1}\alpha_2^{-1}\alpha_1^{-1} \qquad \tilde{\varphi}_7(\beta_2) = \alpha_1\alpha_2\beta_2\beta_3\alpha_3^{-1}$$
$$\tilde{\varphi}_7(\alpha_3) = \beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\alpha_3^{-1}\alpha_1\beta_1\alpha_1^{-1} \qquad \tilde{\varphi}_7(\beta_3) = \alpha_1\alpha_2\beta_2\alpha_3^{-1}\alpha_1\beta_1\alpha_1^{-1}.$$

Then, we have the following:

Theorem 1. (1) *The classes* φ_i *of* $\tilde{\varphi}_i$ *in* $\mathrm{Out}^+(\pi_1(X))$ *generate a subgroup* H *of* Γ_3, *which is isomorphic to* $PSL_2(\boldsymbol{F}_7)$.

(2) *Moreover, if* X *is the Klein curve, then* H *is conjugate to the image of* ι.

Outline of the proof. (1) First note that $H \neq \{1\}$, because the action of H on the homology group $H_1(X, \boldsymbol{Z})$ is not trivial. By direct computation using (3), we have

(4) $\tilde{\varphi}_2^2 = \tilde{\varphi}_3^3 = \tilde{\varphi}_7^7 = 1, \quad \tilde{\varphi}_2\tilde{\varphi}_3\tilde{\varphi}_7 = 1,$
$(\tilde{\varphi}_7\tilde{\varphi}_3\tilde{\varphi}_2)^4 = [\text{conjugation by } \alpha_2\beta_2^{-1}\alpha_2^{-1}\beta_1].$

For example,

$$\begin{aligned}\tilde{\varphi}_3^2\cdot\beta_3 &= (\alpha_2^{-1}\beta_2\alpha_2\alpha_1\alpha_3\alpha_1^{-1}\alpha_2^{-1}\beta_2^{-1}\alpha_2)(\alpha_3\alpha_1^{-1}\beta_1^{-1}\alpha_1) \\ &\quad\times(\alpha_1^{-1}\beta_1\alpha_1\alpha_3^{-1}\alpha_2^{-1}\beta_2\alpha_2\beta_1^{-1}\alpha_1)(\alpha_1^{-1}\beta_1\alpha_1\alpha_3^{-1}) \\ &\quad\times(\beta_1\alpha_2^{-1}\beta_2^{-1}\alpha_2\beta_2\alpha_2)(\alpha_2^{-1}\beta_3^{-1}\alpha_3\alpha_1^{-1}\alpha_2^{-1}\beta_2^{-1}\alpha_2) \\ &\quad\times(\alpha_2^{-1}\beta_2\alpha_2\alpha_1\alpha_3^{-1}\alpha_1^{-1}\alpha_2^{-1}\beta_2^{-1}\alpha_2)(\alpha_2^{-1}\beta_2\alpha_2\alpha_1\alpha_3^{-1}\beta_3\alpha_2) \\ &= \alpha_2^{-1}\beta_2\alpha_2(\alpha_1\beta_1\alpha_2^{-1}\beta_2^{-1}\alpha_2\beta_2\beta_3^{-1}\alpha_3^{-1}\beta_3)\alpha_2 \\ &= \alpha_2^{-1}\beta_2\alpha_2\beta_1\alpha_1\alpha_3^{-1}\alpha_2,\end{aligned}$$

hence

$$\begin{aligned}\tilde{\varphi}_3^3\cdot\beta_3 &= (\alpha_3\alpha_1^{-1}\beta_1^{-1}\alpha_1)(\alpha_1^{-1}\beta_1\alpha_2^{-1}\beta_2^{-1}\alpha_2\alpha_3\alpha_1^{-1}\beta_1^{-1}\alpha_1) \\ &\quad\times(\alpha_1^{-1}\beta_1\alpha_1\alpha_3^{-1})(\alpha_2^{-1}\beta_2\alpha_2\alpha_1\alpha_3\alpha_1^{-1}\alpha_2^{-1}\beta_2^{-1}\alpha_2) \\ &\quad\times(\alpha_2^{-1}\beta_2\alpha_2\alpha_1\alpha_3^{-1}\beta_3\alpha_2)(\alpha_2^{-1}\beta_2^{-1}\alpha_2^{-1}\beta_2\alpha_2\beta_1^{-1})(\alpha_1^{-1}\beta_1\alpha_1\alpha_3^{-1}) \\ &= \beta_3.\end{aligned}$$

From (4) we obtain

(5) $\varphi_2^2 = \varphi_3^3 = \varphi_7^7 = \varphi_2\varphi_3\varphi_7 = (\varphi_7\varphi_3\varphi_2)^4 = 1$

in $\mathrm{Out}^+(\pi_1(X))$. Since (5) is the presentation of

$PSL_2(\boldsymbol{F}_7)$ (see [2, p. 96]), there is a surjective map

$$PSL_2(\boldsymbol{F}_2) \twoheadrightarrow H.$$

The group $PSL_2(\boldsymbol{F}_7)$ is simple, and the map is an isomorphism.

(2) To see that H is the automorphism group of a Riemann surface, it is enough to recall the Nielsen realization problem, which was positively solved in [3]:

Theorem of Kerckhoff. *For any finite subgroup G of Γ_g, there is a compact Riemann surface R of genus g such that*

$$G \subset \mathrm{Aut}(R) \subset \Gamma_g. \qquad \square$$

This theorem shows that there exists a Riemann surface R of genus 3 with $H \subset \mathrm{Aut}(R)$. On the other hand, $\# \mathrm{Aut}(R) \leq 168 = \# H$ by the Hurwitz inequality. Consequently $H = \mathrm{Aut}(R)$. It is classically known (see [5, Th. 2.17]) that the Klein curve is the unique compact Riemann surface of genus 3 such that $\mathrm{Aut}(R) \simeq PSL_2(\boldsymbol{F}_7)$. Thus we have proved Theorem 1. $\square$
Details of the proof and the geometric picture of the automorphisms will appear somewhere else.

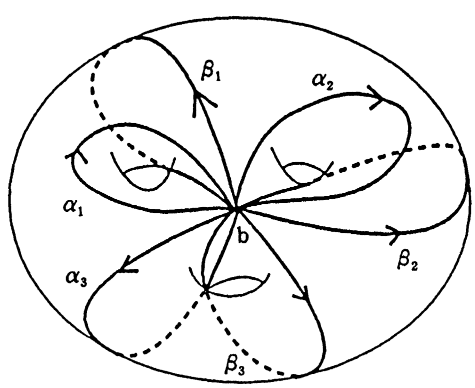

Fig. 1

Acknowledgements. I would like to thank Professor Takayuki Oda for suggesting the problem, and for valuable discussions and encouragements. I also wish to thank Dr. Takuya Konno for careful reading of the manuscript.

References

[1] J. S. Birman: Braids, Links, and Mapping Class Groups. Princeton Univ. Press, Princepton, NJ (1974).

[2] H. S. M. Coxeter and W. O. J. Moser: Generators and Relations for Discrete Groups. Springer Verlag, Berlin and New York (1972).

[3] S. P. Kerckhoff: The Nielsen realization problem. Ann. Math., **117**, 235–265 (1983).

[4] F. Klein: Über die Transformation siebenter Ordnung der elliptischen Functionen. Gesammelte Math. Abhandlungen, Band III, Springer-Verlag, Berlin and New York, pp. 90–136 (1923).

[5] A. Kuribayashi: On analytic families of compact Riemann surfaces with non-trivial automorphisms. Nagoya Math. J., **28**, 119–165 (1966).

[6] A. M. Macbeath: Generators of the Linear Fractional Groups. Proc. Symp. in Pure Math., Amer. Math. Soc., Providence, RI, vol. 12, pp. 14–32 (1968).

[7] W. Magnus: Noneuclidean Tesselations and Their Groups. Academic Press, New York (1970).