Determination of the Harish-Chandra C-function for SU (n, 1) and its Application to the Construction of the Composition Series

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Abstract: The Harish-Chandra C-function for $SU(n, 1)$ is explicitly determined. By this expression, the composition series of the nonunitary principal series is obtained.

1. Introduction. In [1], Harish-Chandra gave an asymptotic expansion of the generalized spherical function on a reductive Lie group $G = KAN$ of the Harish-Chandra class in terms of exponential functions on A and gave the integral expression of the coefficient of the leading term (so-called the Harish-Chandra C-function) of the above expansion.

The Harish-Chandra C-function plays an essential role in studying harmonic analysis of homogeneous spaces of semisimple Lie groups. For instance, it coincides with the restriction of the standard intertwining operator to K-types and it deeply relates with the Plancherel measure. Also we can get some information from it about reducibility of the induced representations.

Though computing explicit expressions of the Harish-Chandra C-functions has been an important problem so far, it is solved in only special cases.

In [3], Johnson and Wallach constructed the composition series for the spherical principal series of the rank one classical groups. To get the result, they obtained the explicit expression of the restriction of the intertwining operator to K-types for the class one representations, which is the component of the Harish-Chandra C-function with respect to the trivial representation of M. In [3, 4] Johnson and Wallach also showed the unitarizability and square-integrability of the irreducible components of the spherical principal series.

Later Kraljević (cf. [7]), Klimyk and Gavrilik (cf. [5]) gave the composition series of the full nonunitary principal series for $SU(n, 1)$. However in their arguments, they used not the expression of the Harish-Chandra C-function but the structure of the K-spectrum.

In this paper, using the method as in [3], we will give the explicit formula of the Harish-Chandra C-function for $SU(n, 1)$ and the composition series obtained in [7] as its application. Concerning the unitarizability and square-integrability, we will discuss in another paper.

2. Notation and preliminaries. We use the standard notation $\boldsymbol{Z}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ for the set of integers, real numbers and complex numbers respectively. Let $\boldsymbol{Z}^+$ be the set of nonnegative integers.

Let $G = SU(n, 1)$ $(n \geq 2)$ and $K = S(U(n) \times U(1))$. Then K is a maximal compact subgroup of G. Define the analytic subgroups A, N and $\bar{N}$ by

$$A = \left\{ \begin{pmatrix} \cosh t & & \sinh t \\ & I_{n-1} & \\ \sinh t & & \cosh t \end{pmatrix} : t \in \boldsymbol{R} \right\},$$

$$N = \left\{ \begin{pmatrix} \frac{3-F}{2} & -\frac{\sqrt{2}}{2} z^* & \frac{F-1}{2} \\ \frac{\sqrt{2}}{2} z & I_{n-1} & -\frac{\sqrt{2}}{2} z \\ \frac{1-F}{2} & -\frac{\sqrt{2}}{2} z^* & \frac{1+F}{2} \end{pmatrix} : z \in \boldsymbol{C}^{n-1}, \right.$$

$$\left. u \in \boldsymbol{R},\ F = 1 + \frac{1}{2} \sum_{i=1}^{n-1} |z_i|^2 + \sqrt{-1}u \right\},$$

$$\bar{N} = \left\{ \begin{pmatrix} \frac{3-F}{2} & -\frac{\sqrt{2}}{2} z^* & \frac{1-F}{2} \\ \frac{\sqrt{2}}{2} z & I_{n-1} & \frac{\sqrt{2}}{2} z \\ \frac{F-1}{2} & \frac{\sqrt{2}}{2} z^* & \frac{1+F}{2} \end{pmatrix} : z \in \boldsymbol{C}^{n-1}, \right.$$

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$$u \in \boldsymbol{R},\ F = 1 + \frac{1}{2}\sum_{i=1}^{n-1} |z_i|^2 + \sqrt{-1}u \Bigg\},$$

where I_{n-1} denotes the unit matrix of order $n-1$ and z^* denotes the conjugate transpose of $z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \end{pmatrix}$. We denote by $\bar{n}(z, u)$ the element in the definition of $\bar{N}$. We identify $\bar{N}$ with $\boldsymbol{C}^{n-1} \times \boldsymbol{R}$ and define the Haar measure $d\bar{n}$ on $\bar{N}$ by $(n-1)!/\pi^n dz d\bar{z} du$, $dz d\bar{z}$ denoting $dz_1 d\bar{z}_1 \ldots dz_{n-1} d\bar{z}_{n-1}$. Let M be the centralizer of A in K and $\mathfrak{a}$ be the Lie algebra of A. Then they are given by

$$M = \left\{ \begin{pmatrix} u & & \\ & X & \\ & & u \end{pmatrix} : X \in U(n-1),\ u \in \boldsymbol{C},\ u^2 \det X = 1 \right\},$$

$$\mathfrak{a} = \{tH : t \in \boldsymbol{R}\}, \text{ where } H = \begin{pmatrix} & & & & 1 \\ & & & 0 & \\ & & \cdot & & \\ & 0 & & & \\ 1 & & & & \end{pmatrix}.$$

For any $g \in G$, let $g = \kappa(g) \exp H(g) n(g)$ be the Iwasawa decomposition of g, where $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$, $n(g) \in N$. Put $w = \operatorname{diag}(-1, -1, 1, \ldots, 1) \in K$. Then w is a representative of the nontrivial element of the Weyl group of G.

The complex dual space $\mathfrak{a}_c^*$ of $\mathfrak{a}$ can be identified with $\boldsymbol{C}$ under the correspondence $\lambda \in \mathfrak{a}_c^* \to \lambda(H) \in \boldsymbol{C}$. Let ρ be the rho function of G. Then ρ is identified with the integer n.

We denote by $\hat{K}$ and $\hat{M}$ the sets of equivalence classes of irreducible unitary representations of K and M respectively. For $\tau \in \hat{K}$ and $\sigma \in \hat{M}$, let $[\tau : \sigma]$ denote the multiplicity of σ occurring in $\tau|_M$. In the case of $SU(n, 1)$, it is known (cf. [6]) that $[\tau : \sigma] = 0$ or 1 for any $\tau \in \hat{K}$ and $\sigma \in \hat{M}$. We write $\hat{M}(\tau)$ for the set of all $\sigma \in \hat{M}$ such that $[\tau : \sigma] = 1$.

For $\tau \in \hat{K}$, the Harish-Chandra C-function associated with τ is defined by the following integral:

$$C_\tau(\lambda) = \int_{\bar{N}} \tau(\kappa(\bar{n}))^{-1} e^{-(\lambda+\rho)(H(\bar{n}))} d\bar{n},\ (\lambda \in \mathfrak{a}_c^*).$$

Let $\tau \in \hat{K}$ and $\sigma \in \hat{M}(\tau)$. We write V_τ and H_σ for the representation spaces of τ and σ, respectively. Choose $P_\sigma(\tau) \in \operatorname{Hom}_M(V_\tau, H_\sigma)$ such that $P_\sigma(\tau)P_\sigma(\tau)^* = \sigma(1)$, the asterisk denoting the adjoint operator. Then because $P_\sigma(\tau)C_\tau(\lambda) \in \operatorname{Hom}_M(V_\tau, H_\sigma)$ and $\dim \operatorname{Hom}_M(V_\tau, H_\sigma) = 1$, there exists a constant $C_\tau(\sigma : \lambda)$ such that $P_\sigma(\tau)C_\tau(\lambda) = C_\tau(\sigma : \lambda)P_\sigma(\tau)$. It is known that the function $\lambda \to C_\tau(\sigma : \lambda)$ becomes a meromorphic function. The function $C_\tau(\sigma : \lambda)$ is called the Harish-Chandra C-function associated with τ and σ. We note that the relationship between the intertwining operator and the Harish-Chandra C-function was discussed in [11]. Moreover, in [12] it was shown that the Harish-Chandra C-function associated with τ and σ can be written as a quotient of products of the classical Γ functions.

3. Representations of K and M. As usual, we parametrize $\hat{K}$ and $\hat{M}$ with their highest weights. According to [6], they are parametrized as follows:

$$\hat{K} = \left\{(t_1, \ldots, t_n) \in \left(\frac{1}{n+1}\boldsymbol{Z}\right)^n : t_i - t_{i+1} \in \boldsymbol{Z}^+\ (i = 1, \ldots, n-1)\right\},$$

$$\hat{M} = \left\{(s_1, \ldots, s_{n-1}) \in \left(\frac{1}{n+1}\boldsymbol{Z}\right)^{n-1} : s_i - s_{i+1} \in \boldsymbol{Z}^+\ (i = 1, \ldots, n-2)\right\}.$$

For $\tau = (t_1, \ldots, t_n) \in \hat{K}$ and $\sigma = (s_1, \ldots, s_{n-1}) \in \hat{M}$, we put $|\tau| = \sum_{i=1}^n t_i$ and $|\sigma| = \sum_{i=1}^{n-1} s_i$. In[6], $\hat{M}(\tau)$ can be written as follows:

$$\hat{M}(\tau) = \{(s_1, \ldots, s_{n-1}) \in \hat{M} : t_i - s_i \in \boldsymbol{Z}^+,\ s_i - t_{i+1} \in \boldsymbol{Z}^+\ (i = 1, \ldots, n-1)\}.$$

On the other hand, the irreducible unitary representations of $SU(n)$ can be constructed in terms of the Young diagram (cf. [13]). For $\tau = (t_1, \ldots, t_n) \in \hat{K}$, put $T(\tau) = (t_1 - t_n, \ldots, t_{n-1} - t_n)$. Let $V_{T(\tau)}$ and $\rho_{T(\tau)}$ denote the subspace of the $|\tau| - nt_n$ fold tensor product of $\boldsymbol{C}^n$ defined by the Young diagram $T(\tau)$ and the action of $SU(n)$ on $V_{T(\tau)}$ induced from the natural one of $SU(n)$ on $\boldsymbol{C}^n$, respectively. Multiplying the character of the center of K by $\rho_{T(\tau)}$, we can construct the irreducible unitary representations of K and show that any elements in $\hat{K}$ can be realized by using this method.

This realization is useful because when restricting $\tau \in \hat{K}$ to M, the decompositions of $V_{T(\tau)}$ into the M-irreducible components can be explicitly written. Thus we can write the explicit expression of the highest weight vector of σ embedded into $V_{T(\tau)}$ by means of the tensor products of the standard basis for $\boldsymbol{C}^n$.

4. Expression of the Harish-Chandra C-function. Let $\tau=(t_1,\ldots,t_n)\in\hat{K}$ and $\sigma=(s_1,\ldots,s_{n-1})\in\hat{M}(\tau)$. Put $\tau(\sigma)=(s_1,\ldots,s_{n-1},s_{n-1})\in\hat{K}$. For computing the Harish-Chandra C-function $C_\tau(\sigma:\lambda)$ associated with τ and σ, we use the explicit expression of $C_{\tau(\sigma)}(\sigma:\lambda)$ and the recursion formula of $C_\tau(\sigma:\lambda)$ with respect to τ.

For getting the explicit expression of $C_{\tau(\sigma)}(\sigma:\lambda)$, we calculate the integral in the definition of the Harish-Chandra C-function. Let v denote the highest weight vector of σ embedded into $V_{T(\tau(\sigma))}$. As described in Section 3, we can write the explicit form of $\tau(\sigma)(\kappa(\bar{n}(z,u)))v$. If $\bar{n}(z,u)\in\bar{N}$ is as in Section 1, we have

$$\tau(\sigma)(\kappa(\bar{n}(z,u)))^{-1}v=\left(\frac{\bar{F}}{|F|}\right)^{-|\tau(\sigma)|-s_{n-1}}\prod_{i=1}^{n-2}\left(\frac{2-F_{i+1}}{\bar{F}}\right)^{s_i-s_{i+1}}v+\text{other terms},$$

where $F_i=F-|z_i|^2-\cdots-|z_{n-1}|^2$ and other terms denote a sum of the lower weight vectors.

Applying the integral formula in [9], we have the following proposition.

Proposition 1. *Retain the above notation. We have the following expression*:

$$C_{\tau(\sigma)}(\sigma:\lambda)=\frac{(n-1)!2^{-\lambda+n}\Gamma(\lambda)}{\prod_{j=1}^{n-1}\left(\frac{\lambda+n+|\sigma|}{2}-j+s_j\right)\Gamma\left(\frac{\lambda+n-|\sigma|}{2}-s_{n-1}\right)\Gamma\left(\frac{\lambda-n+|\sigma|}{2}+1+s_{n-1}\right)}.$$

Let $(\pi_{\sigma,\lambda},\mathscr{H}^{\sigma,\lambda})$ $(\sigma\in\hat{M},\ \lambda\in\mathfrak{a}_c^*)$ be the principal series representation of G induced from the representation $\sigma\otimes\lambda\otimes 1$ of MAN and $H\in\mathfrak{a}$ be as in Section 1. For $v\in V_\tau$, put $\tilde{v}(k)=P_\sigma(\tau)(\tau(k)^{-1}v)\in\mathscr{H}^{\sigma,\lambda}$. For getting the recursion formula, we calculate $\pi_{\sigma,\lambda}(H)\tilde{v}(\sigma\mid\tau)$, where $v(\sigma\mid\tau)$ denotes the highest weight vector of σ embedded into $V_{T(\tau)}$. We note that in the case of class one representation $\sigma=1$, $v(1\mid\tau)$ can be written as a hypergeometric function (cf. [4]). Using this fact, Johnson and Wallach calculated $\pi_\lambda(H)\tilde{v}(1\mid\tau)$ in their paper. For general case, Thieleker [10] gave a formula for computing $\pi_{\sigma,\lambda}(H)\varphi$, where $\varphi\in\mathscr{H}^{\sigma,\lambda}$. We note that Mamiuda [8] specialized this formular to the case of the principal series representation.

Let $A(w,\sigma,\lambda)$ be the standard intertwining operator. According to the arguments in [11], we have

$$A(w,\sigma,\lambda)\tilde{v}(\sigma\mid\tau)=(-1)^{|\tau|-|\tau(\sigma)|+s_1-s_{n-1}}C_\tau(\sigma:\lambda)\tilde{v}(\sigma\mid\tau).$$

For $\tau=(t_1,\ldots,t_n)\in\hat{K}$, let $\chi_i\tau=(t_1,\ldots,t_{i-1},t_i+1,t_{i+1},\ldots,t_n)$. By using the Lemma 1 in [10] and the expression of $v(\sigma\mid\tau)$, we have the following proposition.

Proposition 2. *Let $\tau=(t_1,\ldots,t_n)\in\hat{K}$ and $\sigma=(s_1,\ldots,s_{n-1})\in\hat{M}(\tau)$. Assume that $\chi_i\tau\in\hat{K}$ and $[\chi_i\tau:\sigma]=1$. Then we have the following recursion formula*:

$$(\lambda+n+|\sigma|-2i+2+2t_i)C_{\chi_i\tau}(\sigma:\lambda)=(\lambda-n-|\sigma|+2i-2-2t_i)C_\tau(\sigma:\lambda).$$

Combining proposition 1 and proposition 2, we have the following theorem.

Theorem 3. *Let $\tau=(t_1,\ldots,t_n)\in\hat{K}$ and $\sigma=(s_1,\ldots,s_{n-1})\in\hat{M}(\tau)$. Then we have*

$$C_\tau(\sigma:\lambda)=\frac{(n-1)!2^{-\lambda+n}\Gamma(\lambda)\prod_{j=1}^{n-1}\Gamma\left(\frac{\lambda-n-|\sigma|}{2}+j-s_j\right)\prod_{j=1}^{n-1}\Gamma\left(\frac{\lambda+n+|\sigma|}{2}-j+s_j\right)}{\prod_{j=1}^{n}\Gamma\left(\frac{\lambda-n-|\sigma|}{2}+j-t_j\right)\prod_{j=1}^{n}\Gamma\left(\frac{\lambda+n+|\sigma|}{2}-j+1+t_j\right)}.$$

Remark. When $n=1$, putting $|\sigma|=0$, we see that the result coincides with the expression for $SU(1,1)$ obtained by Johnson [2].

5. Determination of the composition series. We first note that the composition series of the nonunitary principal series for $SU(n,1)$ was already determined by Kraljević [7] or Klimyk and Gavrilik [5]. We shall show that the information on zeros of the Harish-Chandra C-function can be used to determine the composition series.

Let $\sigma=(s_1,\ldots,s_{n-1})\in\hat{M}$ and $\lambda\in\boldsymbol{R}$ such that $\lambda>0$. For simplicity, put $h_j=(\lambda-n-|\sigma|)/2+j-s_j$ and $k_j=(\lambda+n+|\sigma|)/2-j+s_j$. We choose $a,b=0,1,\ldots,n-1$ satisfying the following conditions:

$$h_1<\cdots<h_a\le 0<h_{a+1}<\cdots<h_{n-1},$$
$$k_1>\cdots>k_b>0\ge k_{b+1}>\cdots>k_{n-1}.$$

Then it is clear that $a\le b$. We define the subsets R_σ and Z_σ of $\boldsymbol{R}$ as follows:

$$R_\sigma=\{\lambda>0:\lambda+n+|\sigma|+2s_{n-1}\in 2\boldsymbol{Z}\},$$
$$Z_\sigma=\{\lambda\in R_\sigma:h_a\neq 0\text{ or }k_{b+1}\neq 0\}.$$

We denote by $\Lambda(\sigma)$ the set of $\tau\in\hat{K}$ satisfying $[\tau:\sigma]=1$.

Let $\lambda\in R_\sigma$. Using the expression of $C_\tau(\sigma:\lambda)$, we see that the zeros of $C_\tau(\sigma:\lambda)$ coincide with the ones of the following function:

$$\frac{1}{\Gamma\left(\frac{\lambda-n-|\sigma|}{2}+a+1-t_{a+1}\right)\Gamma\left(\frac{\lambda+n+|\sigma|}{2}-b+t_{b+1}\right)}.$$

In the following discussion, we assume that $s_0 = \infty$ and $s_n = -\infty$. We cosider the following cases.

(1) $h_a = 0$ and $k_{b+1} = 0$.

It is obvious that $C_\tau(\sigma:\lambda) \neq 0$ for all $\tau \in \Lambda(\sigma)$. Therefore $\pi_{\sigma,\lambda}$ is irreducible.

(2) $h_a = 0$ and $k_{b+1} \neq 0$.

Put $S^\sigma_{a,-}(b) = \{\tau = (t_1,\dots,t_n) \in \Lambda(\sigma) : s_{b+1} \leq t_{b+1} \leq a+b-n-|\sigma|-s_a\}$ and let $\mathscr{H}^\sigma_{a,-}(b)$ be the sum of the representation spaces corresponding to the elements in $S^\sigma_{a,-}(b)$. Then $C_\tau(\sigma:\lambda)$ has zero of order 1 for $\tau \in S^\sigma_{a,-}(b)$. Therefore $\pi_{\sigma,\lambda}$ is reducible.

(3) $h_a \neq 0$ and $k_{b+1} = 0$.

Put $S^\sigma_{b,+}(a) = \{\tau = (t_1,\dots,t_n) \in \Lambda(\sigma) : a+b-n-|\sigma|-s_{b+1}+2 \leq t_{a+1} \leq s_a\}$ and let $\mathscr{H}^\sigma_{b,+}(a)$ be the sum of the representation spaces corresponding to the elements in $S^\sigma_{b,+}(a)$. Then $C_\tau(\sigma:\lambda)$ has zero of order 1 for $\tau \in S^\sigma_{b,+}(a)$. Therefore $\pi_{\sigma,\lambda}$ is reducible.

(4) $h_a \neq 0$ and $k_{b+1} \neq 0$.

We define the subsets $S^\sigma_{\lambda,-}(b)$ and $S^\sigma_{b,+}(a)$ of $\Lambda(\sigma)$ as follows:

$$S^\sigma_{\lambda,-}(b) = \Big\{\tau = (t_1,\dots,t_n) \in \Lambda(\sigma) : s_{b+1} \leq t_{b+1} \leq -\frac{\lambda+n+|\sigma|}{2}+b\Big\},$$

$$S^\sigma_{\lambda,+}(a) = \Big\{\tau = (t_1,\dots,t_n) \in \Lambda(\sigma) : \frac{\lambda-n-|\sigma|}{2}+a+1 \leq t_{a+1} \leq s_a\Big\}.$$

Put $S^\sigma_\lambda(a, b) = S^\sigma_{\lambda,-}(b) \cap S^\sigma_{\lambda,+}(a)$. We note that if $a = b$ then $S^\sigma_\lambda(a, b) = \emptyset$. Let $\mathscr{H}^\sigma_{\lambda,-}(b)$, $\mathscr{H}^\sigma_{\lambda,+}(a)$ and $\mathscr{H}^\sigma_\lambda(a, b)$ be the sum of the representation spaces corresponding to the elements in $S^\sigma_{\lambda,-}(b)$, $S^\sigma_{\lambda,+}(a)$ and $S^\sigma_\lambda(a, b)$ respectively. Then $C_\tau(\sigma:\lambda)$ has zero of order 1 for $\tau \in (S^\sigma_{\lambda,-}(b) \cup S^\sigma_{\lambda,+}(a)) \setminus S^\sigma_\lambda(a, b)$ and zero of order 2 for $\tau \in S^\sigma_\lambda(a, b)$. Therefore $\pi_{\sigma,\lambda}$ is reducible.

Remark. In the case of (4), $\pi_{\sigma,\lambda}$ leaves $\mathscr{H}^\sigma_{\lambda,-}(b)$ and $\mathscr{H}^\sigma_{\lambda,+}(a)$ invariant.

Thus we obtain the following theorems.

Theorem 4. *If $\lambda \in Z_\sigma$ then $\pi_{\sigma,\lambda}$ is reducible.*

Theorem 5. *Retain the above notation and note that $a \leq b$. The composition series of $\pi_{\sigma,\lambda}$ ($\lambda \in Z_\sigma$) can be written as follows:*

(1) *$h_a = 0$ and $k_{b+1} \neq 0$.*
$\mathscr{H}^{\sigma,\lambda} \supset \mathscr{H}^\sigma_{a,-}(b) \supset \{0\}$.

(2) *$h_a \neq 0$ and $k_{b+1} = 0$.*
$\mathscr{H}^{\sigma,\lambda} \supset \mathscr{H}^\sigma_{b,+}(a) \supset \{0\}$.

(3) *$h_a \neq 0$ and $k_{b+1} \neq 0$ and $a = b$.*
$\mathscr{H}^{\sigma,\lambda} \supset \mathscr{H}^\sigma_{\lambda,-}(a) + \mathscr{H}^\sigma_{\lambda,+}(a) \supset \mathscr{H}^\sigma_{\lambda,-}(a) \supset \{0\}$,
$\mathscr{H}^{\sigma,\lambda} \supset \mathscr{H}^\sigma_{\lambda,-}(a) + \mathscr{H}^\sigma_{\lambda,+}(a) \supset \mathscr{H}^\sigma_{\lambda,+}(a) \supset \{0\}$.

(4) *$h_a \neq 0$ and $k_{b+1} \neq 0$ and $a < b$.*
$\mathscr{H}^{\sigma,\lambda} \supset \mathscr{H}^\sigma_{\lambda,-}(b) + \mathscr{H}^\sigma_{\lambda,+}(a) \supset \mathscr{H}^\sigma_{\lambda,-}(b) \supset \mathscr{H}^\sigma_\lambda(a, b) \supset \{0\}$,
$\mathscr{H}^{\sigma,\lambda} \supset \mathscr{H}^\sigma_{\lambda,-}(b) + \mathscr{H}^\sigma_{\lambda,+}(a) \supset \mathscr{H}^\sigma_{\lambda,+}(a) \supset \mathscr{H}^\sigma_\lambda(a, b) \supset \{0\}$.

Finally we will describe the reducibility of $\pi_{\sigma,0}$. In this case, $C_\tau(\sigma:\lambda)$ has no zero at $\lambda = 0$. Put $B(w, \sigma, \lambda) = A(w, \sigma, \lambda)/\Gamma(\lambda)$ (cf. [3]). We choose $\sigma \in \hat{M}$ such that $n + |\sigma| + 2s_{n-1} \in 2\boldsymbol{Z}$. Recall the definitions of k_j, h_j and choose $a \in \boldsymbol{Z}^+$ such that $h_a \leq 0 < h_{a+1}$. If $h_a = 0$ then $C_\tau(\sigma:\lambda)$ has pole at $\lambda = 0$, that is, zero of the Plancherel measure. In this case, because the partial intertwining operator $B(w, \sigma, \lambda)$ has no zero, we see that $\pi_{\sigma,\lambda}$ is irreducible. If $h_a \neq 0$ then $C_\tau(\sigma:\lambda)$ is neither zero nor pole. However using the expression of $B(w, \sigma, \lambda)$, we can find the reducible components of $\mathscr{H}^{\sigma,0}$. This is the case of Theorem 5(3). Therefore we have the following theorem.

Theorem 6. *Retain the notation in Theorem 5. If $h_a \in \boldsymbol{Z}\setminus\{0\}$ then $\pi_{\sigma,0}$ is reducible and the composition series of $\mathscr{H}^{\sigma,0}$ is given as follows:*
$\mathscr{H}^{\sigma,0} \supset \mathscr{H}^\sigma_{0,-}(a) \supset \{0\}$,
$\mathscr{H}^{\sigma,0} \supset \mathscr{H}^\sigma_{0,+}(a) \supset \{0\}$.

References

[1] Harish-Chandra: On the Theory of the Eisenstein Integral. Lecture Notes in Mathematics. vol. 266, Springer-Verlag, Berlin-Heidelberg-New York, pp. 123–149 (1972).

[2] K. Johnson: Functional Analysis on $SU(1, 1)$. J. Func. Anal., **14**, 346-363 (1974).

[3] K. Johnson and N. R. Wallach: Composition series and intertwining operators for the spherical principal series. Bull. Amer. Math. Soc., **78**, 1053-1059 (1972).

[4] K. Johnson and N. R. Wallach: Composition series and intertwining operators for the spherical principal series I. Trans. Amer. Math. Soc., **229**, 137–173 (1977).

[5] A. U. Klimyk and A. M. Gavrilik: The representations of the group $U(n, 1)$ and $SO_0(n, 1)$. ITP–76-39E Institute for Theoretical Physics Kiev (1976) (preprint).

[6] K. Kraljević: Representations of the universal

covering group of the group $SU(n, 1)$, Glasnik. Mat. Ser. Ⅲ **8(28)**, 23–72 (1973).

[7] K. Kraljević: On representations of the group $SU(n, 1)$. Trans. Amer. Math. Soc., **221**, 433–448 (1976).

[8] M. Mamiuda: Infinitesinal operators of the principal series representations, Seminar Reports of Unitary Representation. no. 1, pp. 54–61 (1981) (in Japanese).

[9] J. Sekiguchi: On Harish-Chandra's C-function, Seminar Reports of Unitary Representation. no. 1, pp. 68–114 (1981) (in Japanese).

[10] E. Thieleker: On the quasi-simple irreducible representations of the Lorentz groups. Trans. Amer. Math. Soc., **179**, 465–505 (1973).

[11] N. R. Wallach: Harmonic Analysis on Homogeneous Spaces. Marcel Dekker, New York (1973).

[12] D. A. Vogan and N. R. Wallach: Intertwining Operators for Reductive Groups. Adv. in Math., **82**, 203–243 (1990).

[13] H. Weyl: The Classical Groups, their Invariants and Representations. 2nd ed., Princeton Univ. Press, Princeton N. J. (1946).