Orbits in the Flag Variety and Images of the Moment Map for $U(p, q)$

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1. Introduction. Let $G_{\boldsymbol{R}}$ be a real classical group. For simplicity, we consider only the case that Cartan subgroups of $G_{\boldsymbol{R}}$ are connected. It is known that there is a correspondence between irreducible representation of $G_{\boldsymbol{R}}$ with a given infinitesimal character and the orbits of a flag of the complexification of $G_{\boldsymbol{R}}$ on which the complexification K of a maximal compact subgroup of $G_{\boldsymbol{R}}$ operates (for example [7]). Essentially, a parametrization of the orbits is given by Matsuki-Oshima [5]. In this paper, we will consider the case $G_{\boldsymbol{R}} = U(p, q)$ and get an algorithm which gives representatives of orbits for the parametrization and images of the moment map of the conormal bundles of orbits. For a closed orbit, the image is the associated variety of a representation [1]. By the method in this paper, we get representatives from parameters directly. The proof will appear in another paper. Garfinkle gave another algorithm in [3]. By her algorithm we can also get signed Young diagram from a parameter of K-orbits in the flag variety. It remains to examine how two algorithms agree.

The argument in this paper can be applied to the case $G_{\boldsymbol{R}} = Sp(p, q)$, etc. The concerning results will appear elsewhere.

Notation 1.1. Let $\boldsymbol{N}$ denote the set of positive integers; $\boldsymbol{N} = \{1,2,\dots\}$. For $n \in \boldsymbol{N}$, let an $n \times n$ matrix $E_{ij}(1 \le i, j \le n)$ denote the matrix unit which has 1 for the (i, j)-entry and 0 for other entries. Let an n-column vector e_i be the vector which has 1 for the i-th entry and 0 for other entries. For a matrix A, let A_{st} be the (s, t)-entry of A. Let $\mathrm{Mat}(m, n)$ be the set of $m \times n$-matrices over $\boldsymbol{C}$. Let $I_n \in \mathrm{Mat}(n, n)$ be the identity matrix. For an $A \in \mathrm{Mat}(n, n)$ and a subset $\{i(1), i(2), \dots, i(m)\}$ of $\{1, \dots, n\}$, we denote by $A_{(i(1),i(2),\dots,i(m))}$ an $m \times m$-matrix whose (s, t)-entry is $A_{i(s)i(t)}$. Let $\#S$ denote cardinality of the finite set S. For m vectors $\{g_1, \dots, g_m \mid g_i \in \boldsymbol{C}^n\}$ $(m < n)$ let $\langle g_1, \dots, g_m\rangle$ be vector space spanned by $\{g_1, \dots, g_m\}$. Let $\mathfrak{S}_n$ be the set of permutations of $\{1, \dots, n\}$.

2. A symbolic parametrization of K-orbits. Let $G_{\boldsymbol{R}}$ be a real classical Lie group with Lie algebra $\mathfrak{g}_{\boldsymbol{R}}$, G the complexification of $G_{\boldsymbol{R}}$, θ a Cartan involution of $\mathfrak{g}_{\boldsymbol{R}}$. Let $\mathfrak{g}_{\boldsymbol{R}} = \mathfrak{k}_{\boldsymbol{R}} + \mathfrak{p}_{\boldsymbol{R}}$ be the Cartan decomposition corresponding to θ, $\mathfrak{k}$ the complexification of $\mathfrak{k}_{\boldsymbol{R}}$, K the analytic subgroup of G for $\mathfrak{k}$, and B a Borel subgroup of G. Although we restrict ourselves to the case $G_{\boldsymbol{R}} = U(p, q)$ in the main body of the paper, a preliminary discussion holds for an arbitrary linear connected reductive Lie group $G_{\boldsymbol{R}}$.

In this section we recall a symbolic parametrization of K-orbits in X.

Let $n = p + q$. We realize the indefinite unitary group $G_{\boldsymbol{R}} = U(p, q)$ as the group of matrices g in $GL(n, \boldsymbol{C})$ which leave invariant the Hermitian from of the signature (p, q)

$$x_1\overline{x_1} + \cdots + x_p\overline{x_p} - x_{p+1}\overline{x_{p+1}} - \cdots - x_n\overline{x_n},$$

i.e.,

$$U(p, q) = \left\{ g \in GL(n, \boldsymbol{C}) \mid {}^t g \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \bar{g} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \right\}.$$

We fix a Cartan involution θ of $G_{\boldsymbol{R}}$:

$$\theta : x \mapsto \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} x \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Definition 2.1 (Clan) (see [5]). An *indication* for $U(p, q)$ is an ordered set $(c_1 \dots c_n)$ of n symbols satisfying the following four conditions.

1. For every $1 \le i \le n$, c_i is $+$, $-$, or an element of $\boldsymbol{N}$.
2. If $c_i \in \boldsymbol{N}$, then there exists a unique $j \ne i$ with $c_j = c_i$, i.e.,
   $\#\{i \mid c_i = a\} = 0$ or 2 for any $a \in \boldsymbol{N}$.
3. The difference between numbers of $+$ and $-$ in an indication $(c_1 \dots c_n)$ coincides with the difference of signatures of the Hermitian form defining the group $G_{\boldsymbol{R}}$:

$\# \{i \mid c_i = +\} - \# \{i \mid c_i = -\} = p - q.$

4. If $c_i > 1$, then there exists some j such that $c_j = c_i - 1$.

We define an equivalence relation between two indications as follows. Two indications $(c_1 \ldots c_n)$ and $(d_1 \ldots d_n)$ are regarded as equivalent if and only if there exists a permutation $\sigma \in \mathfrak{S}_m$, $m := \max\{d_i \in \boldsymbol{N}\}$ such that

$$ci = \begin{cases} \sigma(d_i) & \text{if } d_i \in \boldsymbol{N} \\ + & \text{if } d_i = + \\ - & \text{if } d_i = - \end{cases}.$$

for all $1 \leq i \leq n$. A *clan* is an equivalence class of the indications with respect to the equivalence relation. For example, $(2\,2\,1 + 1 -) = (1\,1\,2 + 2 -)$ as a clan. We denote the set of clans for $U(p, q)$ by $\mathscr{C}(U(p, q))$. By abuse of notation sometimes we represent a clan γ by an indication belonging to the clan γ.

Theorem 2.2 [5]. *Clans in $\mathscr{C}(U(p, q))$ parametrize K-orbits in the flag variety X.*

Definition 2.3 (Standard indication). If an indication $(c_1 \ldots c_n)$ of a clan satisfies the following condition, we call it *standard.*

If $c_i = c_j = a \in \boldsymbol{N}$ for $i < j$, $c_s = c_t = b \in \boldsymbol{N}$ for $s < t$, and $i < s$, then $a < b$.

Obviously, every clan has a unique standard indication.

Example 2.4. The set $\mathscr{C}(U(2,2))$ consists of 21 clans:

$$\left\{\begin{matrix} + + - - , & + - + - , & + - - + , & - + + - \\ - + - + , & - - + + , & 1\,1 + - , & 1\,1 - + \\ + 1\,1 - , & - 1\,1 + , & + - 1\,1 , & - + 1\,1 \\ 1 + 1 - , & 1 - 1 + , & + 1 - 1 , & - 1 + 1 \\ 1 + - 1 , & 1 - + 1 , & 1\,1\,2\,2 , & 1\,2\,1\,2 \\ 1\,2\,2\,1 & & & \end{matrix}\right\}.$$

Here $(+ + - -)$, for example, is denoted by $+ + - -$, for simplicity.

3. Representatives of K-orbits in X. In this section we will give an element $g \in G$ such that the flag $x = gB$ corresponds to a clan.

Notation 3.1. Let V_+ and V_- be a p and q-dimensional vector subspaces of $\boldsymbol{C}^n$ spanned by $\{e_1, \ldots, e_p\}$ and $\{e_{p+1}, \ldots, e_n\}$ respectively.

Definition 3.2 (Signed clan). A *signed clan* of a clan $\gamma = (c_1 \ldots c_n)$ is an ordered set $(d_1 \ldots d_n)$ of n symbols $+$, $-$, a_+ and a_- for some $a \in \boldsymbol{N}$ satisfying the following two conditions.

1. If $c_i = +$, then $d_i = +$. If $c_i = -$, then $d_i = -$.
2. If $c_i = c_j = a$ for some $a \in \boldsymbol{N}$, then $(d_i, d_j) = (a_+, a_-)$ or (a_-, a_+).

Example 3.3. There are four signed clans for $(1\,1\,2\,2)$:

$$(1_+\,1_-\,2_+\,2_-),\ (1_+\,1_-\,2_-\,2_+),$$
$$(1_-\,1_+\,2_+\,2_-), \text{ and } (1_-\,1_+\,2_-\,2_+).$$

Definition 3.4. For a signed clan $\delta = (d_1 \ldots d_n)$, we say a signature of d_i is *plus* if $d_i = +$ or a_+, *minus* if $d_i = -$ or a_-.

Definition 3.5. For a signed clan $\delta = (d_1 \ldots d_n)$, we have three maps σ_+, σ_-, and σ such that

$$\sigma_+(i) = \sigma_+(\delta, i) := \# \{s \mid s \leq i \text{ and the signature of } d_s \text{ is plus}\},$$
$$\sigma_-(i) = \sigma_-(\delta, i) := \# \{s \mid s \leq i \text{ and the signature of } d_s \text{ is minus}\} + p,$$
$$\sigma(i) = \sigma(\delta, i) := \begin{cases} \sigma_+(i) & \text{if the signature of } d_i \text{ is plus,} \\ \sigma_-(i) & \text{if the signature of } d_i \text{ is minus.} \end{cases}$$

for $1 \leq i \leq n$.

Example 3.6. For a signed clan $(+ - 1_+ + 2_-1_- - 2_+)$, we have

$$\sigma_+ = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 1\,1\,2\,3\,3\,3\,3\,4 \end{pmatrix},$$
$$\sigma_- = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 4\,5\,5\,5\,6\,7\,8\,8 \end{pmatrix}, \text{ and}$$
$$\sigma = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 1\,5\,2\,3\,6\,7\,8\,4 \end{pmatrix},$$

for a signed clan $(+ - 1_+ + 2_+1_- - 2_-)$, we have

$$\sigma_+ = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 1\,1\,2\,3\,4\,4\,4\,4 \end{pmatrix},$$
$$\sigma_- = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 4\,5\,5\,5\,5\,6\,7\,8 \end{pmatrix}, \text{ and}$$
$$\sigma = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 1\,5\,2\,3\,4\,6\,7\,8 \end{pmatrix},$$

by Definition 3.5. Here $\sigma = \begin{pmatrix} 1 & 2 & \ldots \\ i_1 & i_2 & \ldots \end{pmatrix}$ means $\sigma(1) = i_1$, $\sigma(2) = i_2$, and so on.

We give representatives for clans.

Theorem 3.7. *For $\gamma \in \mathscr{C}(U(p, q))$ and a signed clan $\delta = (d_1 \ldots d_n)$ of γ, the matrix $g(\delta) = (g_1\,g_2 \ldots g_n)$ is a representative of Q_γ. Here $g_i \in V$ are column vectors defined as follows.*

- *If $c_i = \pm$, then $g_i = e_{\sigma(i)}$.*
- *If $c_i = a_+$, $c_j = a_-$, then*

$$g_i = \frac{1}{\sqrt{2}}(e_{\sigma(i)} + e_{\sigma(j)}) \text{ and } g_j = \frac{1}{\sqrt{2}}(-e_{\sigma(i)} + e_{\sigma(j)}).$$

Definition 3.8. We define a function $l : \gamma \in \mathscr{C}(U(p, q))$ to $\boldsymbol{Z}$:

$$l(\gamma) = \textstyle\sum_{c_i = c_j \in \boldsymbol{N}, i<j} (j - i - \# \{a \in \boldsymbol{N} \mid c_s = c_t = a \text{ for } s < i < t < j\}).$$

Proposition 3.9. *For $\gamma \in \mathscr{C}(U(p, q))$, we can give the dimension and codimension of the orbit $Q_\gamma = Kg(\delta)B$:*

$$\dim Q_\gamma = l(\gamma) + \frac{1}{2}(p(p-1) + q(q-1)),$$

$$\operatorname{codim} Q_\gamma = pq - l(\gamma).$$

4. Images of the moment map. In this section we give the image of the moment map of a fiber of the conormal bundle of the K-orbit for each clan.

By identifying the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ by means of a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$, the image of the moment map which is a subspace of $\mathfrak{g}^*$ is identified with a vector subspace of $\mathfrak{g}$.

Let $\gamma = (c_1 \ldots c_n)$ be an element of $\mathscr{C}(U(p, q))$. Fix a signed clan δ of γ. The matrix $g(\delta)$ is the representative given in Theorem 3.7. We will describe an algorithm to obtain the image $g(\delta)\mathfrak{b}^\perp g(\delta)^{-1} \cap \mathfrak{p}$ of the moment map of a fiber at $g(\delta)$ of the conormal bundle of $Q_\gamma = Kg(\delta)B$. Here $\mathfrak{b}^\perp$ is a subalgebra of $\mathfrak{g}$ that is orthogonal to $\mathfrak{b}$:

$$\mathfrak{b}^\perp = \{b \in \mathfrak{g} \mid \beta(b, b') = 0 \text{ for all } b' \in \mathfrak{b}\},$$

where $\beta(x, y)$ is trace of ${}^t xy$ on $\mathfrak{g}$:

$$\beta(x, y) = \operatorname{tr}{}^t xy \quad \text{for } x, y \in \mathfrak{g}.$$

We regard $\sigma \in \mathfrak{S}_n$ as an element of $GL(n, \boldsymbol{C})$ such that $\sigma(e_i) = e_{\sigma(i)}$ for all $1 \le i \le n$, i.e.,

$$\sigma = (e_{\sigma(1)} \ldots e_{\sigma(n)}) \in GL(n, \boldsymbol{C}).$$

Then $\mathfrak{S}_p \times \mathfrak{S}_q$ is a subgroup of K. The following theorem gives an image of the moment map.

Theorem 4.1. *For a clan $\gamma = (c_1 \ldots c_n) \in \mathscr{C}(U(p, q))$, fix a signed clan $\delta = (d_1 \ldots d_n)$ of γ. Let representative $g := g(\delta)$ be given by Theorem 3.7 and $x = gB \in Q_\gamma$. For the following vector subspace $\mathrm{Dri}(\delta)$ of $\mathfrak{g}$, we have*

$$\mu(T^*_{Q_\gamma}X)_x = g\mathfrak{b}^\perp g^{-1} \cap \mathfrak{p} = \{\sigma Y \sigma^{-1} \mid Y \in \mathrm{Dri}(\delta)\}$$
$$= \{Y_{(\sigma^{-1}(1), \ldots, \sigma^{-1}(n))} \mid Y \in \mathrm{Dri}(\delta)\}.$$

$$\mathrm{Dri}(\delta) := \{Y \in \mathfrak{g} \mid Y \text{ satisfies the following conditions}\}.$$

1. *If c_i and c_j have the same sign, then $Y_{ij} = Y_{ji} = 0$.*
2. *If $c_i = c_j \in \boldsymbol{N}$, then $Y_{ij} = Y_{ji} = 0$.*
3. *Let $d_i = a_+$, $d_j = a_-$, $d_s = b_+$, $d_t = b_-$, $d_l = +$, $d_m = -$.*
   - (a) $Y_{\min(l,m), \max(l,m)} = 0$
   - (b) *If $\min(i, j) < l < \max(i, j)$, then $Y_{jl} = Y_{lj} = 0$.*
   - (c) *If $\min(i, j) < m < \max(i, j)$, then $Y_{im} = Y_{mi} = 0$.*
   - (d) *If $l < \min(i, j)$, then $Y_{lj} = 0$.*
   - (e) *If $m < \min(i, j)$, then $Y_{mi} = 0$.*
   - (f) *If $\max(i, j) < l$, then $Y_{jl} = 0$.*
   - (g) *If $\max(i, j) < m$, then $Y_{im} = 0$.*
   - (h) *If $\max(i, j) < \min(s, t)$, then $Y_{it} = Y_{js} = 0$.*
   - (i) *If $\min(i, j) < \min(s, t) < \max(i, j) < \max(s, t)$, then $Y_{it} = Y_{js} = 0$, $Y_{ti} - Y_{sj} = 0$.*
   - (j) *If $\min(i, j) < \min(s, t) < \max(s, t) < \max(i, j)$, then $Y_{it} = Y_{ti} = Y_{js} = Y_{sj} = 0$.*

Corollary 4.2. *The image of the moment map $\mu(T^*_{Q_\gamma}X)$ is the K-orbit of $\sigma \cdot \mathrm{Dri}(\delta) \cdot \sigma^{-1}$*, i.e.,

$$\mu(T^*_{Q_\gamma}X) = \{k \cdot Y \mid k \in K,\ Y \in \sigma \cdot \mathrm{Dri}(\delta) \cdot \sigma^{-1}\}.$$

Remark 4.3. For a representative $g(\delta)$ of Q_γ given in Theorem 3.7, we have

$$\mathrm{Dri}(\delta) = \{\sigma^{-1} Y \sigma \mid Y \in g(\delta)\mathfrak{b}^\perp g(\delta)^{-1} \cap \mathfrak{p}\}$$
$$= \{Y_{(\sigma(1), \ldots, \sigma(n))} \mid Y \in g(\delta)\mathfrak{b}^\perp g(\delta)^{-1} \cap \mathfrak{p}\}.$$

5. Signed Young diagrams. In this section we give tables of signed Young diagrams for clans of $U(2,1)$ and $U(2,2)$ by way of examples. A *signed Young diagram* is a Young diagram in which every box is labelled with a $+$ or $-$ sign in such a way that signs alternate across rows and they need not alternate down columns. Two signed Young diagrams are regarded as equal if and only if one can be obtained from the other by interchanging rows of equal length. For $G_{\boldsymbol{R}} = U(p, q)$, nilpotent K-orbits in $\mathfrak{p}$ are parametrized by signed Young diagram which has p boxes labeled $+$ and q boxes labeled $-$ (see [2]).

An element A of $\mathfrak{p}$ satisfies $AV_+ \subset V_-$, $AV_- \subset V_+$.

Definition 5.1. The signed Young diagram of a nilpotent K-orbit is defined as follows. We remark that a signed Young diagram is determined by number of boxes labeled $+$ and number of boxes labeled $-$ of each column.

For an element A of an orbit, the signed Young diagram of the orbit satisfies the following conditions.

$$\#\{(s, j) \mid (s, j)\text{-box is labeled with } + \text{ sign and } j \le i\} = \dim(\ker(A^i|_{V_+}))$$

and

$$\#\{(s, j) \mid (s, j)\text{-box is labeled with } - \text{ sign and } j \le i\} = \dim(\ker(A^i|_{V_-}))$$

for all $1 \le i \le n$.

Proposition 5.2. *Under the conditions of Theorem 3.7, we put*

$$V'_+ := {}^t\sigma V_+ = (e_{\sigma(1)} \ldots e_{\sigma(n)})^{-1} V_+$$
$$= \langle e_{\sigma^{-1}(1)}, \ldots, e_{\sigma^{-1}(p)} \rangle,$$
$$V'_- := {}^t\sigma V_- = (e_{\sigma(1)} \ldots e_{\sigma(n)})^{-1} V_-$$

$$= \langle e_{\sigma^{-1}(p+1)}, \ldots, e_{\sigma^{-1}(n)} \rangle .$$

Then, we have

$$\dim(\ker(Y^i|_{V'_+})) = \dim(\ker(A^i|_{V_+})) \text{ and}$$
$$\dim(\ker(Y^i|_{V'_-})) = \dim(\ker(A^i|_{V_-}))$$

for a generic element $Y \in \mathrm{Dri}(\delta)$ *and a generic element* $A \in \mu(T^*_{Q_r}X)_x$.

Example 5.3.

A case of $G_{\boldsymbol{R}} = U(2,1)$.

<table>
<tr><th>Clan</th><th>Signed Young diagram</th></tr>
<tr><td>+ − +</td><td>$\boxed{+}\boxed{-}\boxed{+}$</td></tr>
<tr><td>+ + −</td><td rowspan="2">$\boxed{-}\boxed{+}$
$\boxed{+}$</td></tr>
<tr><td>+ 1 1</td></tr>
<tr><td>− + +</td><td rowspan="2">$\boxed{+}\boxed{-}$
$\boxed{+}$</td></tr>
<tr><td>1 1 +</td></tr>
<tr><td>1 + 1</td><td>$\boxed{+}$
$\boxed{+}$
$\boxed{-}$</td></tr>
</table>

A case of $G_{\boldsymbol{R}} = U(2,2)$.

<table>
<tr><th>Clan</th><th>Signed Young diagram</th><th>Clan</th><th>Signed Young diagram</th></tr>
<tr><td>+ − + −</td><td>$\boxed{-}\boxed{+}\boxed{-}\boxed{+}$</td><td>− + − +</td><td>$\boxed{+}\boxed{-}\boxed{+}\boxed{-}$</td></tr>
<tr><td>+ − − +</td><td rowspan="3">$\boxed{+}\boxed{-}\boxed{+}$
$\boxed{-}$</td><td>− + + −</td><td rowspan="3">$\boxed{-}\boxed{+}\boxed{-}$
$\boxed{+}$</td></tr>
<tr><td>+ − 1 1</td><td>− + 1 1</td></tr>
<tr><td>1 1 − +</td><td>1 1 + −</td></tr>
<tr><td>+ + − −</td><td rowspan="2">$\boxed{-}\boxed{+}$
$\boxed{-}\boxed{+}$</td><td>− − + +</td><td rowspan="2">$\boxed{+}\boxed{-}$
$\boxed{+}\boxed{-}$</td></tr>
<tr><td>+ 1 1 −</td><td>− 1 1 +</td></tr>
<tr><td>1 + 1 −</td><td rowspan="3">$\boxed{-}\boxed{+}$
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$\boxed{-}$</td><td>1 − 1 +</td><td rowspan="3">$\boxed{+}\boxed{-}$
$\boxed{+}$
$\boxed{-}$</td></tr>
<tr><td>+ 1 − 1</td><td>− 1 + 1</td></tr>
<tr><td>1 + − 1</td><td>1 − + 1</td></tr>
<tr><td>1 1 2 2</td><td rowspan="2">$\boxed{+}\boxed{-}$
$\boxed{-}\boxed{+}$</td><td rowspan="2">1 2 2 1</td><td rowspan="2">$\boxed{+}$
$\boxed{+}$
$\boxed{-}$
$\boxed{-}$</td></tr>
<tr><td>1 2 1 2</td></tr>
</table>

The associated variety of the representation for a clan contains the nilpotent orbit for the signed Young diagram.

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