Minor Summation Formula of Pfaffians and Schur Function Identities

By Masao ISHIKAWA (*) and Masato WAKAYAMA (**,****)

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1. Introduction. In the paper [1], we exploited a minor summation formula of Pfaffians. The prototype of this formula is found in [6]. The merit of our formula is that, by taking various antisymmetric matrices, we obtain considerably various formulas on the summations of minors of a given rectangular matrix. Our motivation was in the use of the enumerative combinatorics and combinatorial representation theory. (See [9].) We are expecting the utility of this formula on various objects in this area. Particularly we think that the applications on the Schur function identities are important and we studied them intensively in [2]. There we obtained new proof of the formulas which are usually called Littlewood’s formulas. Typical examples of Littlewood’s formulas are the followings.

(1.1) \[ \sum_{\lambda=(\lambda_1,\ldots,\lambda_m)} (-1)^{\frac{m(m+1)}{2}} s_{\lambda}(x_1,\ldots, x_m) \]

where \( s_{\lambda}(x_1,\ldots, x_m) \) is a Schur function.

2. Basic notation and a summation formula. In the paper [1] we exploited a minor summation formula of Pfaffians. Now we briefly review this formula.

Let \( r, m, n \) be positive integers such that \( r \leq m, n \). Let \( T \) be an arbitrary \( m \times n \) matrix. For two sequences \( i = (i_1, \ldots, i_r) \) and \( k = (k_1,\ldots, k_m) \), let \( T_{i\cdots i}^{k\cdots k} \) denote the submatrix of \( T \) obtained by picking up the rows and columns indexed by \( i \) and \( k \), respectively.

Assume \( m \leq n \) and let \( B \) be an arbitrary \( n \times n \) antisymmetric matrix, that is, \( B = (b_{ij}) \) satisfies \( b_{ij} = -b_{ji} \). As long as \( B \) is a square antisymmetric matrix, we write \( B_i = B_{k\cdots k} \) for \( B_i = B_{i\cdots i}^{k\cdots k} \) in abbreviation. One of the main results in [1] is the following theorem. (See Theorem 1 of [1].)

**Theorem 2.1.** Let \( m \leq n \) and \( T = (t_{ik}) \) be an arbitrary \( m \times n \) matrix. Let \( m \) be even and \( B = (b_{ij}) \) be any \( n \times n \) antisymmetric matrix with entries \( b_{ij} \). Then

\[ \sum_{\lambda=(\lambda_1,\ldots,\lambda_m)} pf(B_{1\cdots m}) det(T_{1\cdots m}^{\lambda_1\cdots \lambda_m}) = pf(Q), \]

where \( Q \) is the \( m \times m \) antisymmetric matrix defined by \( Q = TB'T \), i.e.,

\[ Q_{ij} = \sum_{1 \leq k \leq m} b_{ij} det(T_{ki}), \quad (1 \leq i, j \leq m). \]

We regard the Pfaffian \( pf(B_{ik}) \) as certain "weights" of the subdeterminants \( det(T_{1\cdots m}^{\lambda_1\cdots \lambda_m}) \). By changing this antisymmetric matrix we obtain a considerably wide variation of the minor summation formula.

Now we review some basic notation. The reader can find these notation in [5]. A weakly decreasing sequence of nonnegative integers \( \lambda = (\lambda_1, \cdots, \lambda_m) \) with \( \lambda_1 \geq \cdots \geq \lambda_m \geq 0 \) is called a partition of \( | \lambda | = \lambda_1 + \cdots + \lambda_m \). The partition
Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition expressed in the Frobenius notation. Let $a$ and $b$ be nonnegative integers such that $a < m$ and $b < m$. There are some $k$ and $l$ such that $a_k > a > a_k + 1$ and $b_l > b > b_{l+1}$. The partition $\lambda \cup (a \mid b)$ is defined by

$$\lambda \cup (a \mid b) = (\alpha_\lambda \cup a, a, a, \ldots, x_m \mid b, b, b, \ldots, x_m).$$

For example, $(421 \mid 310) \cup (0 \mid 2) = (421013210).$

The Schur functions are well-known symmetric functions, which are known as the values of characters of the irreducible polynomial representations of the general linear group on a torus. But, here, we briefly review the definition of the Schur functions. Put

$$T_n(x) = \frac{\prod_{i=1}^{n} (x_i - 1)}{\prod_{i=1}^{n} (1 - x_i)},$$

for some fixed $n$. For a partition $\lambda := (\lambda_1, \ldots, \lambda_m)$, let $l = (l_1, \ldots, l_m) = \lambda + \delta$, where $\delta = (m - 1, m - 2, \ldots, 0)$. So we have $l_1 > l_2 > \cdots > l_m \geq 0$. Put $j_k = n - l_k$ for $1 \leq k \leq m$. Then we set $a_{j_k} = a_{j_k + 1} = \cdots = a_{j_m} = 1$ to be

$$a_{j_{k+1}} = \det(T_{l_1 - j_m}).$$

When $\lambda = 0$, $a_\lambda$ is the famous Vandermonde determinant and equal to the product $\prod_{1 \leq i < j \leq m} (x_i - x_j)$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, the Schur function $s_{\lambda} = s_\lambda(x_1, \ldots, x_m)$ corresponding to $\lambda$ is defined to be

$$s_{\lambda} = a_{j_{k+1}} / a_{\lambda}.$$

(See Chap. 1, Sec. 3 of [5]).

The polynomials defined by $T_n(x) = \cos(n \arccos x)$ are called Chebyshev's polynomials of the first kind, and, on the other hand, the polynomials $U_n(x) = \sin(n \arccos x) / \sqrt{1 - x^2}$ are called Chebyshev's polynomials of the second kind. Both are known to satisfy the same recurrence formula:

$$P_{n+1}(x) = 2xP_n(x) + P_{n-1}(x) = 0.$$

The first few polynomials are easily calculated from the following recursion formula.

$$T_0(x) = 1, \quad U_0(x) = 0,$$

$$\{ T_{n+1}(x) = xT_n(x) + (x^2 - 1)U_n(x) \}
\{ U_{n+1}(x) = T_n(x) + xU_n(x) \}.$$

3. Littlewood type formulas. The following lemma is the key lemma to evaluate the pfaffian we treat.

Lemma 3.1. Let $m$ be a positive integer and put

$$Q_m(x, y) = \frac{(x^m - y^m)^2}{(x - y)(1 - tx)}.$$

Then

$$\text{pf}[Q_m(x_i, x_j)]_{1 \leq i, j \leq 2m} = \prod_{1 \leq i < j \leq 2m} (x_i - x_j)(1 - tx).$$

We fix $T = (x_i^{4m+d-2-j})_{1 \leq i \leq 2m, 0 \leq j \leq 4m+d-2}$ in this section. We assume $d = 2$ for a moment. Let $B = (b_{ij})$ be an antisymmetric matrix defined through the equation below.

$$\sum_{0 \leq k \leq 4m} b_{ki} \begin{vmatrix} x^k & x^l \\ y^k & y^l \end{vmatrix} = - (1 + 2ax + x^2)(1 - 2ay + y^2) \frac{(x^m - y^m)^2}{x - y},$$

If we apply Theorem 2.1 to $Q$ given by the right hand side of this equation, then we obtain the following formula from Lemma 3.1 with $t = 0$.

Proposition 3.1. Let $m$ be a positive integer.

$$\sum_{k=0}^{m} U_{k+1}(a) \sum_{r=0}^{m-k} s_{i_j}(x_1, \ldots, x_m) = \prod_{i=1}^{m} (1 + 2ax + x^2).$$

If we put $x_i = q^{2i}$ in this formula and put $m \to \infty$, then we obtain a (combinatorial) proof of the $q$-expansion formula of Jacobi theta functions $\theta_1$ and $\theta_2$, for example.

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+\frac{1}{2})^2}}{n!} \sin(2n + 1)\pi u$$

$$= 2\pi q^{\frac{1}{2}} \sin \pi u \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2\pi u + q^{4n}),$$

where $q = e^{i\pi \tau}$ (Im $\tau > 0$) and $\prod_{n=1}^{\infty} (1 - q^{2n}).$
Let $m$ be a positive integer and let $B = (b_{ki})_{0 \leq k, l \leq m-1}$ be an antisymmetric matrix of size $m$ in the ordinary means. Set $b_i$ to be the $i$-th row vector of $B$ for $0 \leq i \leq m-1$. The matrix $B$ is said to be (row-)symmetrically proportional if the $(m - 1 - k)$-th row is proportional to the $k$-th. That is to say, there is some $c_k$ such that $b_{m-1-k} = c_k b_k$ or $b_k = c_{m-1-k} b_{m-1-k}$ for each $0 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$. Further $B$ is called row-symmetric if the $b_{m-1-k} = b_k$ for $0 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$, and $B$ is called row-antisymmetric if the $b_{m-1-k} = -b_k$ for $0 \leq k \leq \left\lfloor \frac{m+1}{2} \right\rfloor - 1$.

This notion has importance since it makes us possible to find all the subpfaffians $pf(B_{1...m})$ of $B$. From now on we assume that $B$ is always supposed to be antisymmetric matrix in the ordinary means.

Let $P(x) = a_0 + a_1 x + \cdots + a_d x^d$ be a polynomial of degree $d$. $P(x)$ is said to be symmetric if $a_i = a_{d-i}$ for $0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$, and $P(x)$ is said to be antisymmetric if $a_i = -a_{d-i}$ for $0 \leq i \leq \left\lfloor \frac{d+1}{2} \right\rfloor$. Then we have

**Lemma 3.2.** Let $P(x)$ be a polynomial of degree $d$. Let $B = (b_{ki})_{0 \leq k, l \leq 4m+d-2}$ be the antisymmetric matrix of size $(4m + d - 1)$ which satisfy

$$
\begin{align*}
0 \leq k, l \leq 4m+d-2 & \quad b_{ki} | x^k y^l | = -P(x) P(y) Q_m(x, y).
\end{align*}
$$

The matrix $B$ becomes (row-)symmetrically proportional for all $m$ if and only if $P(x)$ is symmetric or antisymmetric. Further, if the polynomial $P(x)$ is symmetric then $B$ becomes row-symmetric, on the other hand, if $P(x)$ is antisymmetric then $B$ becomes row-antisymmetric.

From now we apply Theorem 2.1 to this $T$ and $B$ given by (3.6). Basically it is possible to find some sort of formula for each antisymmetric matrix $B$ of the formula (3.6) if it is row-symmetric or row-antisymmetric. Here we investigate each formula for small $d$. When $d = 0$, we obtain the formula (1.1) from this argument. If $d = 1$ and $P(x)$ is antisymmetric, we obtain the formula (1.2). It is easy to see that the case of $d = 1$ and $P(x)$ being symmetric reduces to this case. If $d = 2$ and $P(x)$ is antisymmetric, then we obtain the formula (1.3). These are the known Littlewood type formulas. If we assume $d = 2$ and $P(x)$ is symmetric, then we obtain the following theorem.

**Theorem 3.1.** Let $m$ be a positive integer. Then

$$
\sum_{j=(a+1)(a)} (-1) \frac{\eta(p^{(j)})}{2} s_j(x_1, \ldots, x_m)
+ \sum_{k=1}^{m} T_k(a) \sum_{j=(a+1)(a)} (-1) \frac{\eta(p^{(j)})}{2} q(\lambda, k-1) \times s_{j+1}(x_{1, \ldots, x_m})
= \prod_{i=1}^{m} (1 + 2ax_i + x_i^2) \prod_{1 \leq i < j \leq m} (1 - x_i x_j).
$$

If we put $x = q^{\frac{2}{m}}$ in this formula and we use the $q$-expansion formula of Jacobi theta function $\theta_3$, we obtain the following corollary.

**Corollary 3.1.**

$$
\sum_{j=(a+1)(a)} (-1) \frac{\eta(p^{(j)})}{2} q^{\frac{1}{2} \frac{n(j)}{2}} \cdots \frac{1}{1-q^{2a}} \prod_{1 \leq i < j \leq m} (1 - q^{2a}) \prod_{j=2}^{m} (1 - q^{2a})
$$

Let $m$ be a nonnegative integer. Then

$$
\sum_{j=(a+1)(a)} (-1) \frac{\eta(p^{(j)})}{2} q^{\frac{1}{2} \frac{n(j)}{2}} \cdots \frac{1}{1-q^{2a}} \prod_{1 \leq i < j \leq m} (1 - q^{2a}) \prod_{j=2}^{m} (1 - q^{2a})
$$

If $d = 3$ and $P(x)$ is antisymmetric, we obtain the following theorem. The case of $d = 3$ and $P(x)$ being symmetric essentially reduces to this case.

**Theorem 3.2.** Let $m$ be a positive integer. Then

$$
\sum_{j=(a+2)(a)} (-1) \frac{\eta(p^{(j)})}{2} s_j(x_1, \ldots, x_m)
+ \sum_{k=1}^{m} T_k(a) \sum_{j=(a+2)(a)} (-1) \frac{\eta(p^{(j)})}{2} q(\lambda, k-1) \times s_{j+1}(x_{1, \ldots, x_m})
= \prod_{i=1}^{m} (1 + 2ax_i + x_i^2) \prod_{1 \leq i < j \leq m} (1 - x_i x_j).
$$

If $d = 4$ and $P(x)$ is antisymmetric, we obtain the following theorem.

**Theorem 3.3.** Let $m$ be a positive integer. Then

$$
\sum_{j=(a+3)(a)} (-1) \frac{\eta(p^{(j)})}{2} s_j(x_1, \ldots, x_m)
+ \sum_{k=1}^{m} U_{k+1}(a) \sum_{j=(a+3)(a)} (-1) \frac{\eta(p^{(j)})}{2} q(\lambda, k-1) \times s_{j+1}(x_{1, \ldots, x_m})
= \prod_{i=1}^{m} (1 + 2ax_i + x_i^2) \prod_{1 \leq i < j \leq m} (1 - x_i x_j).
$$
\[
\times \{s_{1/4(0)}(x_1, \ldots, x_m) - s_{1/4(2)}(x_1, \ldots, x_m)\} \\
= \prod_{i=1}^{m} (1 + 2ax_i + x_i^2) \prod_{1 \leq i \leq j \leq m} (1 - x_i x_j).
\]

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References


