On Certain Dirichlet Series Obtained by the Product of Eisenstein Series and a Cusp Form

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§1. Let $\mathscr{H}$ be an upper half plane ,$\Gamma = SL_2(\boldsymbol{Z})$ and Γ_∞ be the stabilizer of the cusp $i\infty$ of Γ. The real analytic Eisenstein series $E(z, \alpha)$ is defined by

$$E(z, \alpha) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\mathrm{Im}\gamma z)^\alpha \text{ for } \mathrm{Re}\,\alpha > 1.$$

We put $E^*(z, \alpha) = \xi(2\alpha)E(z, \alpha)$ where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\zeta(s)$ is the Riemann zeta function. It is well known that the function $E^*(z, \alpha)$ has a holomorphic continuation to all α except for simple poles at $\alpha = 0$ and 1 and satisfies the functional equation $E^*(z, \alpha) = E^*(z, 1 - \alpha)$. The Fourier expansion is given by

$$E^*(z, \alpha) = \xi(2\alpha)y^\alpha + \xi(2 - 2\alpha)y^{1-\alpha} + 2\sum_{n \neq 0} |n|^{1/2-\alpha}\sigma_{2\alpha-1}(|n|)y^{1/2}K_{\alpha-1/2}(2\pi|n|y)e^{2\pi inx}.$$

Here, $K_\nu(z)$ denotes the so-called modified Bessel function and $\sigma_\nu(n) = \sum_{d|n} d^\nu$.

In [5], Vinogradov and Takhtadzhyan studied the classical additive divisor problem through the spectral theory of automorphic functions. Namely they showed that the main term of the integral

$$\int_0^\infty \int_0^1 |E^*(z, 1/2)|^2 y^s e^{2\pi ikz} \frac{dxdy}{y^2}$$

is $\pi^{-s}\Gamma(s/2)^4\Gamma(s)^{-1}\sum_{n=1}^\infty d(n)d(n+k)n^{-s}$ and got the growth order of the last Dirichlet series by the spectral theory of automorphic functions.

§2. We consider here the product of the Eisenstein series and a cusp form and derive the corresponding Dirichlet series. Let $f(z)$ be a Maass wave form with the parity ε_f and its Fourier expansion be given by

$$f(z) = \sum_{n \neq 0} \rho(n)y^{1/2}K_{i\kappa}(2\pi|n|y)e^{2\pi inx}.$$

We assume that $\rho(n) = O(|n|^{\eta_0})$ for some $\eta_0 > 0$. Up to now, it is known that $\eta_0 \leq 5/28$. (cf. [1])

For a natural integer k, we define

$$I_k(s\,;\alpha, f) = \int_0^\infty \int_0^1 E^*(z, \alpha) f(z) y^s e^{2\pi ikx} \frac{dxdy}{y^2}.$$

Lemma 1. *Let s be a complex number. If* $\mathrm{Re}\,s$ *is sufficiently large, we have*

$$\begin{aligned} I_k(s\,;\alpha, f) &= (4\pi^s\Gamma(s))^{-1}\Gamma\Big(\frac{s+\alpha-1/2+i\kappa}{2}\Big) \\ &\times \Gamma\Big(\frac{s+\alpha-1/2-i\kappa}{2}\Big)\Gamma\Big(\frac{s-\alpha+1/2+i\kappa}{2}\Big) \\ &\times \Gamma\Big(\frac{s-\alpha+1/2-i\kappa}{2}\Big) \\ &\times \Big\{\sum_{m=1}^\infty \frac{\sigma_{2\alpha-1}(m+k)\rho(m)}{m^{s+\alpha-1/2}} F\Big(\frac{s+\alpha-1/2+i\kappa}{2}, \\ &\qquad \frac{s+\alpha-1/2-i\kappa}{2}\,;s\,;1-\Big(\frac{m+k}{m}\Big)^2\Big) \\ &+ \varepsilon_f \sum_{m=1, m\neq k}^\infty \frac{\sigma_{2\alpha-1}(|m-k|)\rho(m)}{m^{s+\alpha-1/2}} \\ &\times F\Big(\frac{s+\alpha-1/2+i\kappa}{2}, \frac{s+\alpha-1/2-i\kappa}{2}\,;s\,; \\ &\qquad 1-\Big(\frac{m-k}{m}\Big)^2\Big)\Big\} \\ &+ \varepsilon_f\rho(k)\varphi_0(s\,;\alpha), \end{aligned}$$

where $F(\alpha, \beta, \gamma\,;x)$ is the hypergeometric function and

$$\begin{aligned} \varphi_0(s, \alpha) &= \frac{\xi(2\alpha)}{4(\pi k)^{s+\alpha-1/2}} \\ &\times \Gamma\Big(\frac{s+\alpha-1/2+i\kappa}{2}\Big)\Gamma\Big(\frac{s+\alpha-1/2-i\kappa}{2}\Big) \\ &+ \frac{\xi(2-2\alpha)}{4(\pi k)^{s-\alpha+1/2}}\Gamma\Big(\frac{s-\alpha+1/2+i\kappa}{2}\Big) \\ &\times \Gamma\Big(\frac{s-\alpha+1/2-i\kappa}{2}\Big). \end{aligned}$$

This lemma can be shown by the Fourier expansions of $E^*(z, \alpha)$, $f(z)$ and the following integral formula:

$$\begin{aligned} \int_0^\infty K_\nu(ny)K_\mu(my)\,dy &= 2^{s-3}m^{-s-\nu}n^\nu\Gamma(s)^{-1} \\ &\times \Gamma\Big(\frac{s+\mu+\nu}{2}\Big)\Gamma\Big(\frac{s+\mu-\nu}{2}\Big)\Gamma\Big(\frac{s-\mu+\nu}{2}\Big) \\ &\times \Gamma\Big(\frac{s-\mu-\nu}{2}\Big) \\ &\times F\Big(\frac{s+\nu+\mu}{2}, \frac{s+\nu-\mu}{2}\,;s\,;1-(n/m)^2\Big). \end{aligned}$$

(cf. [2] p. 93 (36))

We now introduce a Dirichlet series, the main term of I_k. We put

$$D_k(s\,;\alpha,f)=\sum_{m=1}^{\infty}\frac{\sigma_{2\alpha-1}(m+k)\rho(m)}{m^s}+\varepsilon_f\sum_{m=1,m\neq k}^{\infty}\frac{\sigma_{2\alpha-1}(|m-k|)\rho(m)}{m^s}.$$

This series converges absolutely for $\mathrm{Re}\, s>2\,\mathrm{Re}\,\alpha+\eta_0$ if $\mathrm{Re}\,\alpha\geq 1/2$, and $\mathrm{Re}\, s>1+\eta_0$ if $\mathrm{Re}\,\alpha<1/2$. If we write

$$G(\alpha,\beta\,;\gamma\,;z)=F(\alpha,\beta\,;\gamma\,;z)-1,$$

then we have, by the power series expansion and the integral expression of F,

$$G(\alpha,\beta\,;\gamma\,;z)=\frac{\alpha\beta}{\gamma}z\int_0^1F(\alpha+1,\beta+1\,;\gamma+1\,;zx)dx$$
$$=\frac{\alpha\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}z\int_0^1\int_0^1t^{\beta}(1-t)^{\gamma-\beta-1}\times(1-txz)^{-\alpha-1}dtdx.$$

for $\mathrm{Re}\,\gamma>\mathrm{Re}\,\beta>-1$. So Lemma 1 can be written as follows.

Proposition 1. *The notation being as above, then we have*

$$D_k(s\,;\alpha,f)=4\pi^{s-\alpha+1/2}\Gamma\Big(s-\alpha+\frac{1}{2}\Big)\times\Big(\Gamma\Big(\frac{s+i\kappa}{2}\Big)\Gamma\Big(\frac{s-i\kappa}{2}\Big)\Gamma\Big(\frac{s-2\alpha+1+i\kappa}{2}\Big)\times\Gamma\Big(\frac{s-2\alpha+1-i\kappa}{2}\Big)\Big)^{-1}\Big(I_k\Big(s-\alpha+\frac{1}{2}\,;\alpha,f\Big)-\varepsilon_f\rho(k)\varphi_0\Big(s-\alpha+\frac{1}{2},\alpha\Big)\Big)-R_k(s\,;\alpha,f)$$

where

$$R_k(s\,;a,f)=\sum_{m=1}^{\infty}\frac{\sigma_{2\alpha-1}(m+k)\rho(m)}{m^s}\times G\Big(\frac{s+i\kappa}{2},\frac{s-i\kappa}{2}\,;s-\alpha+\frac{1}{2}\,;1-\Big(\frac{m+k}{m}\Big)^2\Big)+\varepsilon_f\sum_{m=1,m\neq k}^{\infty}\frac{\sigma_{2\alpha-1}(|m-k|)\rho(m)}{m^s}\times G\Big(\frac{s+i\kappa}{2},\frac{s-i\kappa}{2}\,;s-\alpha+\frac{1}{2}\,;1-\Big(\frac{m-k}{m}\Big)^2\Big)$$

and is absolutely convergent for $\mathrm{Re}\, s>2\mathrm{Re}\,\alpha-1+\eta_0$ *if* $\mathrm{Re}\,\alpha\geq 1/2$, *and* $\mathrm{Re}\, s>\eta_0$ *if* $\mathrm{Re}\,\alpha<1/2$. *Furthermore, we have*

$$R_k(s\,;\alpha,f)\ll|\mathrm{Im}\, s|^{\frac{3}{2}}.$$

The last statement can be obtained by the Stirling formula of Γ function.

§3. Let $u_0(z)=\sqrt{3/\pi}$, the constant function, and $u_j(z)$ $j=1,2,\ldots$ be Maass wave forms constituting an orthonormal basis of cusp forms of $L^2(\Gamma\backslash\mathscr{H})$ with eigenvalues $1/4+\kappa_j^2$ of non-Euclidean Laplacian. Let the Fourier expansion of $u_j(z)$ be

$$u_j(z)=\sum_{n\neq0}\rho_j(n)y^{1/2}K_{i\kappa_j}(2\pi|n|y)e^{2\pi inx}.$$

The parity of $u_j(z)$ is denoted by ε_j.

Lemma 2. *For* $\mathrm{Re}\, s>1/2$ *and* $\mathrm{Re}\,\alpha>0$, *we have*

$$I_k(s,\alpha,f)=\frac{1}{4(\pi k)^{s-1/2}}\sum_{j=1}^{\infty}\varepsilon_jA_j(\alpha)\rho_j(k)\times\Gamma\Big(\frac{s-1/2+i\kappa_j}{2}\Big)\Gamma\Big(\frac{s-1/2-i\kappa_j}{2}\Big)+\frac{1}{8\pi(\pi k)^{s-1/2}}\int_{-\infty}^{\infty}A(\alpha,r)\frac{k^{-ir}\sigma_{2ir}(k)}{\xi(1+2ir)}\times\Gamma\Big(\frac{s-1/2+ir}{2}\Big)\Gamma\Big(\frac{s-1/2-ir}{2}\Big)dr.$$

where

$$A_j(\alpha)=\frac{(1+\varepsilon_f\varepsilon_j)\zeta(2\alpha)}{4\pi^{2\alpha}}\Gamma\Big(\frac{\alpha+i(\kappa+\kappa_j)}{2}\Big)\times\Gamma\Big(\frac{\alpha+i(\kappa-\kappa_j)}{2}\Big)\Gamma\Big(\frac{\alpha-i(\kappa+\kappa_j)}{2}\Big)\times\Gamma\Big(\frac{\alpha-i(\kappa-\kappa_j)}{2}\Big)L_j(\alpha),$$

$$A(\alpha,r)=\frac{(1+\varepsilon_f)\zeta(2\alpha)}{4\pi^{2\alpha}\xi(1-2ir)}\Gamma\Big(\frac{\alpha+i(\kappa+r)}{2}\Big)\times\Gamma\Big(\frac{\alpha+i(\kappa-r)}{2}\Big)\Gamma\Big(\frac{\alpha-i(\kappa+r)}{2}\Big)\times\Gamma\Big(\frac{\alpha-i(\kappa-r)}{2}\Big)L(\alpha,r).$$

and $L_j(\alpha)$ *and* $L(\alpha,r)$ *are meromorphically continued functions which are defined by*

$$\sum_{n=1}^{\infty}\frac{\rho(n)\overline{\rho_j(n)}}{n^{\alpha}},\quad\sum_{n=1}^{\infty}\frac{\rho(n)\sigma_{2ir}(n)}{n^{\alpha+ir}}$$

for $\mathrm{Re}\,\alpha>1+2\eta_0$, *respectively.*

By this formula, we can see that $I_k(s\,;\alpha,f)$ is meromorphically continued to all s. If $H_f(s)$ denotes the Hecke series associated to f, we have

$$L(\alpha,r)=\rho(1)\frac{H_f(\alpha+ir)H_f(\alpha-ir)}{\zeta(2\alpha)}.$$

§4. From now on, we assume that $\mathrm{Re}\,\alpha>1+2\eta_0$. We want to know the growth order of $I_k(s-\alpha+1/2\,;\alpha,f)$ when $|\mathrm{Im}\, s|\to\infty$. First we consider the discrete part. By the assumption on α, the series $L_j(\alpha)$ is absolutely convergent, so we estimate it trivially and get

$$A_j(\alpha)\ll\exp\Big(-\frac{\pi}{2}\kappa_j\Big)\kappa_j^{2\mathrm{Re}\alpha-\frac{3}{2}}.$$

Let $s=\sigma+it$ and $t'=t-\mathrm{Im}\,\alpha$. We divide the sum on κ_j, into three parts, namely, $\kappa_j<t'-c\log(t')$, $t'-c\log(t')\leq\kappa_j\leq t'+c\log(t')$ and

$\kappa_j > t' + c\log(t')$ for some constant c and use the method of partial summation. We have

$$\sum_{j>0} \varepsilon_j A_j(\alpha)\rho_j(k)\, \Gamma\Big(\frac{s-\alpha+i\kappa_j}{2}\Big)\Gamma\Big(\frac{s-\alpha-i\kappa_j}{2}\Big)$$

$$\ll \begin{cases} \exp\Big(-\dfrac{\pi}{2}|t|\Big)|t|^{\sigma+\mathrm{Re}\alpha-\frac{1}{2}+\varepsilon} & \text{if } \sigma \geq \mathrm{Re}\,\alpha+1 \\ \exp\Big(-\dfrac{\pi}{2}|t|\Big)|t|^{\frac{1}{2}\sigma+\frac{3}{2}\mathrm{Re}\alpha+\varepsilon} & \text{if } 0<\sigma<\mathrm{Re}\,\alpha+1 \end{cases}$$

for any fixed $\varepsilon > 0$. We note that for the first two sums on κ_j, we use the Kuznetsov's famous result ([4]):

$$\sum_{\kappa_j \leq X} \frac{|\rho_j(k)|^2}{\cosh \pi\kappa_j} = \pi^{-2}X^2 + O(X\log X + Xk^{\varepsilon} + k^{\frac{1}{2}+\varepsilon}).$$

We can estimate the continuous part similarly and see that it is smaller than the discrete one. Estimates for $\varphi_0(s-\alpha+1/2, \alpha)$ and $R_k(s, \alpha)$ are easy. Hence, by Proposition 1, we get

Proposition 2. *Let* $s = \sigma + it$ *and suppose that* $\mathrm{Re}\,\alpha > 1 + 2\eta_0$. *When* $|\mathrm{Im}\, s| \to \infty$, *we have*

$$D_k(s\,;\alpha, f) \ll \begin{cases} |t|^{2\mathrm{Re}\alpha+\frac{1}{2}+\varepsilon} & \text{if } \sigma \geq \mathrm{Re}\,\alpha+1 \\ |t|^{\frac{5}{2}\mathrm{Re}\alpha-\frac{1}{2}\sigma+1+\varepsilon} & \text{if } 0<\sigma<\mathrm{Re}\,\alpha+1 \end{cases}$$

for any $\varepsilon > 0$.

§5. We put $\sigma_1 = 2\alpha + \eta_0 + \varepsilon' - 1$ and $\sigma_2 = \sigma_1 + 1$ $(\varepsilon' > 0)$. The Dirichlet series $D_k(s\,;\alpha)$ is convergent absolutely on $\mathrm{Re}\, s = \sigma_2$. Let $\mathfrak{R}$ be a rectangle with vertices $\sigma_2 - iT$, $\sigma_2 + iT$, $\sigma_1 + iT$ and $\sigma_1 - iT$. Considering the contour integral $\int_{\mathfrak{R}} D_k(s\,;\alpha, f)\frac{x^s}{s}\,ds$ and using Perron's formula with suitable T, we get

Theorem. *Assume that* $\mathrm{Re}\,\alpha > 1 + 2\eta_0$, *then for any* $\varepsilon > 0$, *we have*

$$\sum_{m\leq x, m\neq k} \{\sigma_{2\alpha-1}(m+k) + \varepsilon_f\sigma_{2\alpha-1}(|m-k|)\}\rho(m) = O(x^{2\mathrm{Re}\alpha+\eta_0+\varepsilon-1/(2\mathrm{Re}\alpha+3/2)}).$$

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