A Characterization of Regularly Almost Periodic Minimal Flows

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Abstract: In this paper we shall prove two theorems: Firstly, a minimal flow is regularly almost periodic if and only if it is almost automorphic and the dimension of the set of eigenvalues is 1. Secondly, a minimal flow is pointwise regularly almost periodic if and only if it is equicontinuous and the dimension of the set of eigenvalues is 1.

§1. Introduction. Let X be a metric space with metric d_X. Z, Q, R and C denote the set of integers, rational numbers, real numbers and complex numbers, respectively. A continuous mapping $\pi : X \times R \rightarrow X$ is said to be a *flow on* (*a phase space*) X if π satisfies the following conditions:

(1) $\pi(x, 0) = x$ for $x \varepsilon X$.

(2) $\pi(\pi(x, t), s) = \pi(x, t+s)$

for $x \varepsilon X$ and $t, s \varepsilon R$.

For $A \subset X$ and $B \subset R$, we denote the set $\{\pi(x, t) ; x \varepsilon A, t \varepsilon B\}$ by $\pi(A, B)$. The closure of $A \subset X$ is denoted by $\bar{A}$. For $x \varepsilon X$ we denote the orbit through $x \varepsilon X$ by $O_\pi(X)$, that is, $O_\pi(x) = \pi(x, R)$. $M \subset X$ is called *an invariant set of* π if $O_\pi(x) \subset M$ for each $x \varepsilon M$. The restriction of π to an invariant set M of π is denoted by $\pi | M$. A non-empty compact invariant set $M \subset X$ is said to be *a minimal set of* π if we have $\overline{O_\pi(x)} = M$ for each $x \varepsilon M$. If X is itself a minimal set of π, we say that π is *a minimal flow on* X. π is said to be *equicontinuous* if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_X(\pi(x, t), \pi(y, t)) < \varepsilon$ for $d_X(x, y) < \delta$ and $t \varepsilon R$.

Let π be a minimal flow on a compact metric space X. $x \varepsilon X$ is called *a regularly almost periodic point* if for each $\varepsilon > 0$ there exists an $\alpha > 0$ such that $\pi(x, n\alpha) \varepsilon U_\varepsilon(x)$ for $n \varepsilon Z$, where $U_\varepsilon(x) = \{z \varepsilon X ; d_X(x, z) < \varepsilon\}$. The set of regularly almost periodic points is denoted by $R(\pi)$. If $R(\pi) \neq \phi$, we say that π is *regularly almost periodic*. If $R(\pi) = X$, we say that π is *pointwise regularly almost periodic*. $x \varepsilon X$ is said to be *an almost automorphic point* if $\pi(x, \tau_n) \rightarrow y$ as $n \rightarrow \infty$ for some sequence $\{\tau_n\} \subset R$ implies that $\pi(y, -\tau_n) \rightarrow x$ as $n \rightarrow \infty$. The set of almost automorphic points is denoted by $A(\pi)$. If $A(\pi) \neq \phi$, we say that π is *almost automorphic*. We can easily see that $R(\pi)$ and $A(\pi)$ are invariant sets of π. $\lambda \varepsilon R$ is said to be *an eigenvalue of* π if there exists a continuous mapping $\chi_\lambda : X \rightarrow K = \{\xi \varepsilon C ; |\xi| = 1\}$ such that $\chi_\lambda(\pi(x, t)) = \chi_\lambda(x)\exp(i\lambda t)$ for $x \varepsilon X$ and $t \varepsilon R$. In this case, χ_λ is called *an eigenfunction belonging to* λ. The set of eigenvalues of π is denoted by $\Lambda(\pi)$. We can easily verify that $\Lambda(\pi)$ is a countable subgroup of the additive group R.

$\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n \varepsilon R$ are said to be *rationally independent* if $r_1\alpha_1 + r_2\alpha_2 + \ldots\ldots\ldots + r_n\alpha_n = 0 (r_i \varepsilon Q)$ implies $r_1 = r_2 = \ldots\ldots\ldots = r_n = 0$. We say that a countable subset A of R has *dimension* n if there exist $\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n \varepsilon R$, which are rationally independent, such that we have $a = r_1\alpha_1 + r_2\alpha_2 + \ldots\ldots\ldots + r_n\alpha_n (r_i \varepsilon Q)$ for each $a \varepsilon A$. The dimension of $A \subset R$ is denoted by $\dim A$.

In [4] regularly almost periodic minimal flows are discussed for discrete phase group. In this paper we characterize them for one parameter flows. In section 2 we shall show the following theorems.

Theorem 1. *Let π be a minimal flow on a compact metric space X. Then π is regularly almost periodic if and only if it is almost automorphic and* $\dim \Lambda(\pi) = 1$.

Theorem 2. *Let π be a minimal flow on a compact metric space X. Then π is pointwise regularly almost periodic if and only if it is equicontinuous and* $\dim \Lambda(\pi) = 1$.

§2. Proofs of Theorems. In this section we shall prove Theorems 1 and 2. In order to prove them, we need several propositions.

Let π and ρ be flows on compact metric spaces X and Y, respectively. A continuous mapping $h: X \rightarrow Y$ is said to a *homomorphism from* π *to* ρ if $h(\pi(x, t)) = \rho(h(x), t)$ for $(x, t) \varepsilon X \times R$. Furthermore, if h is a homeomorphism from X onto Y, we say that h is an *isomorphism from* π *to* ρ. In this case, we say that π and ρ are *isomorphic*. The following proposition is well known.

Proposition 2.1. *Let* π *and* ρ *be equicontinuous minimal flows on compact metric spaces* X *and* Y, *respectively. Then* π *and* ρ *are isomorphic if and only if* $\Lambda(\pi) = \Lambda(\rho)$.

Proposition 2.2. *Let* π *and* ρ *be minimal flows on compact metric spaces* X *and* Y, *respectively, and* h *a homomorphism from* π *to* ρ. *Then* $x_0 \varepsilon R(\pi)$ *implies* $h(x_0) \varepsilon R(\rho)$.

Proof. Easy.

Corollary 2.2.1. *Under the assumption in Proposition* 2.2, *if* π *and* ρ *are isomorphic, and if* π *is pointwise regularly almost periodic, then* ρ *is so.*

Proof. Easy.

Let B_U be the set of bounded and uniformly continuous function from R to C. Define a metric in B_U by $d_{B_U}(f, g) = \sup_{t \varepsilon R}\{| f(t) - g(t) |\}$ for $f, g \varepsilon B_U$. Then B_U is a complete metric space. We define a flow η on B_U by $\eta(f, t) = f_t$ for $(f, t) \varepsilon B_U \times R$, where $f_t(s) = f(t+s)$ for $s \varepsilon R$. Then η is an equicontinuous flow on B_U. For $f \varepsilon B_U$, put $O_\eta(f) = \overline{\{f_t\}_{t \varepsilon R}} = H(f)$ and $\eta_f = \eta \mid H(f)$. A set $L \subset R$ is said to be *relatively dense* if there exists a $l > 0$ such that for each $t \varepsilon R$ we have $[t - l, t + l] \cap L \neq \phi$. A complex valued function f is said to be almost periodic if for each $\varepsilon > 0$ there exists a relatively dense subset $A_\varepsilon \subset R$ such that $| f(t+\tau) - f(t) | < \varepsilon$ for $\tau \varepsilon A_\varepsilon$ and $t \varepsilon R$.

Proposition 2.3. *Let* f *be an almost periodic function. Then we have*

(1) $f \varepsilon B_U$ *and* $H(f)$ *is compact.*

(2) η_f *is equicontinuous minimal flow on* $H(f)$.

(3) *For each* $\lambda \varepsilon R$, $\lim_{t \to \infty} \frac{1}{t}\int_0^t f(s)\exp(-i\lambda s)ds$ *exists.*

Put $\Lambda_f = \left\{\lambda \varepsilon R ; \lim_{t \to \infty} \frac{1}{t}\int_0^t f(s)\exp(-i\lambda s) ds \neq 0\right\}$. *Then* $\Lambda(\eta_f) = \tilde{\Lambda}_f$, *where* $\tilde{\Lambda}_f$ *is the least additive subgroup of* R *containing* Λ_f.

Proof. See [1].

Corollary 2.3.1. *Let* π *be an equicontinuous minimal flow on a compact metric space* X *with* $\Lambda(\pi) = \{\lambda_n\}_{n=1}^\infty$, *and* $\sum_{n=1}^\infty | a_n | < \infty (a_n \varepsilon C - \{0\})$. *Put* $f(t) = \sum_{n=1}^\infty a_n \exp(i\lambda_n t)$. *Then* $f(t)$ *is almost periodic, and* π *and* η_f *are isomorphic.*

Proof. Since $\Lambda(\eta_f) = \tilde{\Lambda}_f = \Lambda_f = \{\lambda_n\}_{n=1}^\infty = \Lambda(\pi)$, the corollary follows from Proposition 2.1.

Proposition 2.4. *Let* π *be a minimal flow on a compact metric space* X. *Then* $x \varepsilon R(\pi)$ *implies* $x \varepsilon A(\pi)$, *that is, a regularly almost periodic minimal flow is almost automorphic.*

Proof. See [6], p. 337.

Proposition 2.5. *Let* π *be an almost automorphic minimal flow on a compact metric space* X. *Then there exist an equicontinuous minimal flow* ρ *on* Y *and a homomorphism* h *from* π *to* ρ *such that* $A(\pi) = \{x \varepsilon X ; h^{-1}\{(h(x)\} = \{x\}\}$. *In this case we have* $\Lambda(\pi) = \Lambda(\rho)$. *Furthermore, if* $A(\pi) = X$, *then* π *is equicontinuous.*

Proof. For the first statement, see [7], p. 737. For the second one, see [2], p. 151. The last statement follows from the first one

Proposition 2.6. *Let* π *be a minimal flow on a compact metric space* X. *For* $\alpha > 0$ *and* $x \varepsilon X$, *put* $C_\alpha(x) = \{\pi(x, n\alpha) ; n \varepsilon Z\}$. *If there exists* $\alpha > 0$ *such that* $\overline{C_\alpha(x)} \neq X$, *then* $\Lambda(\pi) \neq \{0\}$.

Proof. See [1].

Proposition 2.7. *Let* π *be a minimal flow on a compact metric space* X. *We assume that* $\overline{C_\alpha(x)} \neq X$ for $x \varepsilon X$ and $\alpha > 0$. *Then there exists* $\tau_\alpha > 0$ *satisfies following conditions*:

(1) $\{s ; \pi(x, s) \varepsilon \overline{C_\alpha(x)}\} = \{n\tau_\alpha\}_{n \varepsilon Z}$.

(2) $\overline{C_{\tau_\alpha}(x)} = \overline{C_\alpha(x)}$.

(3) $y \varepsilon \overline{C_\alpha(x)}$ *implies* $\overline{C_{\tau_\alpha}(y)} = \overline{C_\alpha(y)} = \overline{C_\alpha(x)}$.

(4) $\pi\left(\overline{C_\alpha(x)}, \left[-\frac{\tau_\alpha}{2}, \frac{\tau_\alpha}{2}\right]\right) = X$.

(5) *For* $-\frac{\tau_\alpha}{2} \leqq t_1 < t_2 < \frac{\tau_\alpha}{2}$, *we have* $\pi(\overline{C_\alpha(x)}, t_1) \cap \pi(\overline{C_\alpha(x)}, t_2) = \phi$.

(6) *For* $0 < \varepsilon < \frac{\tau_\alpha}{2}$, $\pi(\overline{C_\alpha(x)}, (-\varepsilon, \varepsilon))$ *is open in* X *and homeomorphic to* $\overline{C_\alpha(x)} \times (-\varepsilon, \varepsilon)$.

Proof. See [1].

Proposition 2.8. *Let* π *be a minimal flow on a compact metric space* X. *If* $x_0 \varepsilon R(\pi)$, *then* $\overline{C_\alpha(x_0)} \neq X$ *for some* $\alpha > 0$. *Furthermore, if* $\overline{C_\alpha(x_0)} \neq X$, *for each neighborhood* $V(x_0)$ *of* x_0, *there exist* $m \varepsilon Z (m > 0)$ *such that* $\pi(x_0, nm\tau_\alpha) \varepsilon \overline{C_\alpha(x_0)} \cap V(x_0)$ *for* $n \varepsilon Z$, *where* τ_α *is the positive number in Proposition* 2.7.

Proof. The first statement is obvious. For $0 < \varepsilon < \frac{\tau_\alpha}{6}$, put $U = V(x_0) \cap \pi(\overline{C_\alpha(x_0)}, (-\varepsilon, \varepsilon))$. Then U is a neighborhood of x_0 by Proposition 2.7. Hence, by the assumption, there exists $\mu > 0$ such that $\pi(x_0, n\mu) \varepsilon U$ for $n \varepsilon Z$. Since $\pi(x_0, \mu) \varepsilon \pi(\overline{C_\alpha(x_0)}, (-\varepsilon, \varepsilon))$, there exist $m \varepsilon Z (m > 0)$ and $\nu \varepsilon R(|\nu| < \varepsilon)$ such that $\mu = m\tau_\alpha + \nu$. We assume $\nu \neq 0$, Choose $l \varepsilon Z(l > 0)$ so that $|l\nu| < \varepsilon$ and $|(l+1)\nu| \geqq \varepsilon$. Since $\varepsilon \leqq (l+1)|\nu| \leqq |l\nu| + |\nu| < 2\varepsilon < \frac{\tau_\alpha}{3} < \frac{\tau_\alpha}{2}$, we have $\pi(\overline{C_\alpha(x_0)}, (l+1)\nu) \cap U = \phi$ by Proposition 2.7. On the other hand, $\pi(x_0, (l+1)\mu) = \pi(x_0, (l+1)(m\tau_\alpha + \nu)) = \pi(\pi(x_0, (l+1)m\tau_\alpha), (l+1)\nu) \varepsilon \pi(\overline{C_\alpha(x_0)}, (l+1)\nu)$. Since $\pi(x_0, (l+1)\mu) \varepsilon U$, this is a contradiction. Consequently, $\mu = m\tau_\alpha$ that is $\pi(x_0, nm\tau_\alpha) \varepsilon U \cap \overline{C_\alpha(x_0)} \subset V(x_0) \cap \overline{C_\alpha(x_0)}$.

Proposition 2.9. *Let π be a regularly almost periodic minimal flow on a compact metric space X. If $x_0 \varepsilon R(\pi)$ and $\overline{C_\alpha(x_0)} \neq X$ $(\alpha > 0)$, then $\overline{C_\alpha(x_0)}$ is 0 dimension at x_0.*

Proof. For any neighborhood $V'(x_0)$ of x_0, we choose a neighborhood $V(x_0)$ of x_0 such that $V(x_0) \subset V'(x_0)$. Then there exists $m \varepsilon Z(m > 0)$ such that $\pi(x_0, nm\tau_\alpha) \varepsilon V(x_0) \cap \overline{C_\alpha(x_0)}$ for $n \varepsilon Z$ by Proposition 2.8. Since $\overline{C_{m\tau_\alpha}(x_0)} \subset V'(x_0) \cap \overline{C_\alpha(x_0)}$, $\overline{C_{m\tau_\alpha}(x_0)}$ is closed in $\overline{C_\alpha(x_0)}$. On the other hand, for sufficient small $\varepsilon > 0$, $\pi(\overline{C_{m\tau_\alpha}(x_0)}, (-\varepsilon, \varepsilon)) \cap V'(x_0)$ is open in X by Proposition 2.7. Hence $\overline{C_{m\tau_\alpha}(x_0)} = \pi(\overline{C_{m\tau_\alpha}(x_0)}, (-\varepsilon, \varepsilon)) \cap \overline{C_\alpha(x_0)}$ is open in $\overline{C_\alpha(x_0)}$. Consequently, $\overline{C_\alpha(x_0)}$ is 0 dimension at x_0.

Corollary 2.9.1. *Let π be a pointwise regularly almost periodic minimal flow on a compact metric space X. Then, if $\overline{C_\alpha(x)} \neq X$ for $x \varepsilon X$ and $\alpha > 0$, then $\overline{C_\alpha(x)}$ is 0 dimension, that is, it is totally disconnected.*

Proof. If $y \varepsilon \overline{C_\alpha(x)}$, then $\overline{C_\alpha(y)} = \overline{C_\alpha(x)}$ by Proposition 2.7. Hence, since $y \varepsilon R(\pi)$, $\overline{C_\alpha(y)}$ is 0 dimension at y. This implies that $\overline{C_\alpha(x)}$ is 0 dimension at every point in $\overline{C_\alpha(x)}$. Hence $\overline{C_\alpha(x)}$ is 0 dimension.

Corollary 2.9.2. *Let π be a regularly almost periodic minimal flow on a compact metric space X. If $x_0 \varepsilon R(\pi)$, then X is 1 dimension at x_0.*

Proof. Choose $\alpha > 0$ so that $\overline{C_\alpha(x_0)} \neq X$. For a sufficient small $\varepsilon > 0$, $\pi(\overline{C_\alpha(x_0)}, (-\varepsilon, \varepsilon))$ is open in X and homeomorphic to $\overline{C_\alpha(x_0)} \times (-\varepsilon, \varepsilon)$ by Proposition 2.7. Hence, since $\overline{C_\alpha(x_0)}$ is 0 dimension at x_0, X is 1 dimension at x_0 ([5], p. 33).

Proposition 2.10. *Let π be an equicontinuous minimal flow on a compact metric space X. If $R(\pi) \neq \phi$, then $R(\pi) = X$, that is, it is pointwise regularly almost periodic.*

Proof. Let $x_0 \varepsilon R(\pi)$. Given $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_X(x, y) < \delta$ and $t \varepsilon R$ implies $d_X(\pi(x, t), \pi(y, t)) < \frac{\varepsilon}{3}$. For $0 < \delta < \frac{\varepsilon}{3}$, there exists $\alpha > 0$ such that $d_X(x_0, \pi(x_0, n\alpha)) < \delta$ for $n \varepsilon Z$. Since π is minimal, for $x \varepsilon X$ there exists $s \varepsilon R$ such that $d_X(x, \pi(x_0, s)) < \delta$. For this α we have

$$d_X(x, \pi(x, n\alpha))$$
$$\leqq d_X(x, \pi(x_0, s)) + d_X(\pi(x_0, s), \pi(\pi(x_0, n\alpha), s))$$
$$+ d_X(\pi(x_0, s), n\alpha), \pi(x, n\alpha)) < \varepsilon$$

Hence $x \varepsilon R(\pi)$, that is, $R(\pi) = X$.

Proposition 2.11. *Let π be an equicontinuous minimal flow on a compact metric space X. If* dim $\Lambda(\pi) = 1$, *then it is pointwise regularly almost periodic.*

Proof. Let $\Lambda(\pi) = \{\lambda_n\}_{n=1}^{\infty}$, where $\lambda_1 = 0$ and $\lambda_n \neq 0$ $(n \geqq 2)$, and $\sum_{n=1}^{\infty} |a_n| < \infty (a_n \varepsilon C - \{0\})$. Put $f(t) = \sum_{n=1}^{\infty} a_n \exp(i\lambda_n t)$ for $t \varepsilon R$. By Corollaries 2.2.1 and 2.3.1 and Proposition 2.10, it is enough to show that f is a regularly almost periodic point of η_f. Since $\dim \Lambda(\pi) = 1$, there exists $\beta > 0$ such that $\lambda_n = \frac{p_n}{q_n}\beta$ for $n = 2, 3, \ldots$, where p_n, $q_n \varepsilon Z$ are prime to each other. Given $\varepsilon > 0$, we choose $N \varepsilon Z(N > 0)$ so that $\sum_{N+1}^{\infty} |a_n| < \frac{\varepsilon}{2}$. Put $\alpha = \frac{2\pi}{\beta} q_2 q_3 \cdots\cdots\cdots q_N$. Since $\exp(i\lambda_k \alpha) = \exp(2\pi i p_k q_2 q_3 \cdots\cdots q_{k-1} q_{k+1} \cdots q_N) = 1$ for $2 \leqq k \leqq N$, we have

$$|f(t) - f_{n\alpha}(t)|$$
$$= |\sum_{k=1}^{\infty} a_k \exp(i\lambda_k t) - \sum_{k=1}^{\infty} a_k \exp(i\lambda_k(t + n\alpha))|$$
$$\leqq |\sum_{k=1}^{N} a_k \exp(i\lambda_k t) - \sum_{k=1}^{N} a_k \exp(i\lambda_k t)(\exp(i\lambda_k \alpha)^n|$$
$$+ |\sum_{K=N+I}^{\infty} a_k \exp(i\lambda_k t) - \sum_{k=N+1}^{\infty} a_k \exp(i\lambda_k(t + n\alpha))|$$

$$< \sum_{k=N+1}^{\infty} |a_k| + \sum_{k=N+1}^{\infty} |a_k| < \varepsilon.$$

Hence f is a regularly almost periodic point of η_f.

Proof of Theorem 1. Assume that $R(\pi) \neq \phi$. Then $A(\pi) \neq \phi$, since $R(\pi) \subset A(\pi)$ by Proposition 2.4. Hence π is almost automorphic. By Propositions 2.6. and 2.8, we have $\Lambda(\pi) \neq \{0\}$. To prove $\dim \Lambda(\pi) = 1$, we assume that there exist $\lambda_1, \lambda_2 \varepsilon \Lambda(\pi)$ which are rationally independent. Let χ_{λ_1} and χ_{λ_2} be eigenfunctions belonging to λ_1 and λ_2, respectively. Define a flow ρ on $T^2 = K \times K$ by $\rho((\xi_1, \xi_2), t) = (\xi_1 \exp(i\lambda_1 t), \xi_2 \exp(i\lambda_2 t))$ for $(\xi_1, \xi_2) \varepsilon T^2$ and $t \varepsilon R$. Then ρ is an equicontinuous minimal flow on T^2. Define a mapping $h : X \to T^2$ by $h(x) = (\chi_{\lambda_1}(x), \chi_{\lambda_2}(x))$. Then h is a homomorphism from π to ρ. Since, if $x_0 \varepsilon R(\pi)$, we have $h(x_0) \varepsilon R(\rho)$ by Proposition 2.2, T^2 is 1 dimension at $h(x_0)$ by Corollary 2.9.2. This is a contradiction, because T^2 is obviously 2 dimension at $h(x_0)$. Hence $\dim \Lambda(\pi) = 1$.

Conversely, we assume that $A(\pi) \neq \phi$ and $\dim \Lambda(\pi) = 1$. Then there exist an equicontinuous minimal flow ρ on Y and a homomorphism h from π to ρ such that $A(\pi) = \{x ; h^{-1}\{(h(x)\}) = \{x\}\}$ by Proposition 2.5. In this case, since $\dim \Lambda(\pi) = \dim \Lambda(\rho) = 1$ and ρ is equicontinuous, ρ is pointwise regularly almost periodic by Proposition 2.11. The restriction of h to $A(\pi)$ is a homeomorphism from $A(\pi)$ to $h(A(\pi))$ with respect to the relative topology, because h is injection and continuous. For $x \varepsilon A(\pi)$, let $V(x)$ be an open neighborhood of x. Then $h(V(x) \cap A(\pi)) = h(V(x)) \cap h(A(\pi))$ is open in $h(A(\pi))$ with respect to the relative topology. Hence there exist an open set U of Y such that $U \cap h(A(\pi)) = h(A(\pi) \cap V(x))$. Since ρ is regularly almost periodic, there exist $\alpha > 0$ such that $\rho(h(x), n\alpha) \varepsilon U (n \varepsilon Z)$. Since $\rho(h(x), n\alpha) = h(\pi(x, n\alpha)) (n \varepsilon Z)$ and $h(A(\pi))$ is an invariant set of ρ, we have $\rho(h(x), n\alpha) \varepsilon U \cap h(A(\pi)) = h(V(x) \cap A(\pi))$. Consequently, $\pi(x, n\alpha) \varepsilon A(\pi) \cap V(x) (n \varepsilon Z)$. This implies $x \varepsilon R(\pi)$. Hence π is regularly almost periodic.

Proof of Theorem 2. We assume that π is pointwise regularly almost periodic. Then $X = R(\pi) \subset A(\pi)$ means $A(\pi) = X$. Hence π is equicontinuous by Proposition 2.5. Furthermore, $\dim \Lambda(\pi) = 1$ follows from Theorem 1. The converse is Proposition 2.11.

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