On the Generators of the Mapping Class Group of a 3-dimensional Handlebody

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Let V be a handlebody of genus $g(\geq 2)$ and let $T = \partial V$. Let $\mathcal{M}_V$, $\mathcal{M}_T$ be the mapping class groups of V, T, respectively. (For the definition of a mapping class group, see [3].) It is well known that $\mathcal{M}_T$ is isomorphic to the outer automorphism group of $\pi_1(T)$.

We have the injection $\nu : \mathcal{M}_V \to \mathcal{M}_T$ by letting the restriction $f \mid T : T \to T$ correspond to each homeomorphism $f : V \to V$.

In this paper we seek the generators of $\nu(\mathcal{M}_V)(\subset \mathcal{M}_T)$ which are as simple as possible as the products of the generators defined by Lickorish in [4].

Let $\alpha_i, \beta_i\ (i = 1, \cdots, g)$, $\nu_i (i = 1, \cdots, g-1)$ be the isotopy classes of the Dehn twists about the simple loops shown in the Fig. 1.

We shall prove the following theorem.

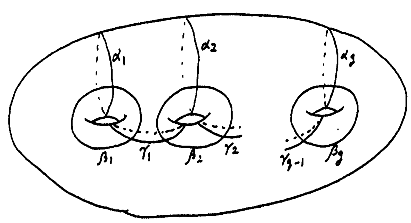

Fig. 1

Theorem. $\nu(\mathcal{M}_V)$ is generated by α_1, $\beta_1\alpha_1^2\beta_1$, $\beta_i\alpha_i\gamma_i\beta_i$, $\beta_{i+1}\alpha_{i+1}\gamma_i\beta_{i+1}\ (i = 1, \cdots, g-1)$.

Proof. Let $a_i, b_i\ (i = 1, \cdots, g)$ be the generators of the fundamental group of the surface T as shown in the Fig. 2.

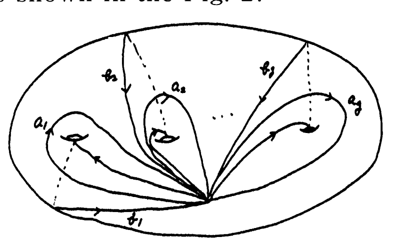

Fig. 2

$(\pi_1(T) \simeq \langle a_i, b_i (i = 1, \cdots, g) \mid s_1^{-1}s_2^{-1} \cdots s_g^{-1} = 1\rangle$, where $s_i = a_i^{-1}b_i^{-1}a_ib_i$, $i = 1, \cdots, g$.)

By Suzuki [1], $\mathcal{M}_V$ is generated by the isotopy classes of τ_1, ω_1, θ_{12}, ξ_{12}, ρ_{12} and ρ. The induced automorphisms of $\pi_1(T)$ are given by:

$$\nu(\tau_1)\ \begin{cases} a_1 \to b_1^{-1}a_1 \\ a_i \to a_i\ (i = 2, \cdots, g) \\ b_i \to b_i (i = 1, \cdots, g), \end{cases}$$

$$\nu(\omega_1)\ \begin{cases} a_1 \to a_1^{-1}s_1^{-1} \\ a_i \to a_i (i = 2, \cdots, g) \\ b_1 \to a_1^{-1}b_1^{-1}a_1 \\ b_i \to b_i (i = 2, \cdots, g) \end{cases}$$

$$\nu(\theta_{12})\ \begin{cases} a_1 \to a_1s_2^{-1}a_1^{-1} \\ a_i \to a_i (i = 2, \cdots, g) \\ b_2 \to a_2b_2a_1^{-1}b_1a_1b_2^{-1}a_2^{-1}b_2 \\ b_i \to b_i (i \neq 2) \end{cases}$$

$$\nu(\xi_{12})\ \begin{cases} a_1 \to b_1a_1b_2^{-1}s_2a_1^{-1}b_1^{-1}a_1 \\ a_2 \to a_2b_2a_1^{-1}b_1^{-1}a_1b_2^{-1} \\ a_i \to a_i (i = 3, \cdots, g) \\ b_i \to b_i (i = 1, \cdots, g), \end{cases}$$

$$\nu(\rho_{12})\ \begin{cases} a_1 \to s_1^{-1}a_2s_1 \\ a_2 \to a_1 \\ a_i \to a_i (i = 3, \cdots, g) \\ b_1 \to s_1^{-1}b_2s_1 \\ b_2 \to b_1 \\ b_i \to b_i (i = 3, \cdots, g) \end{cases}$$

$$\nu(\rho)\ \begin{cases} a_i \to a_{i+1}\ (i = 1, \cdots, g-1) \\ a_g \to a_1 \\ b_i \to b_{i+1} (i = 1, \cdots, g-1) \\ b_g \to b_1 \end{cases}$$

First we observe that each element stated in the theorem is actually an element of $\nu(\mathcal{M}_V)$. By [2], an element of $\mathcal{M}_T$ is in $\nu(\mathcal{M}_V)$ if and only if, by the induced automorphism of $\pi_1(T)$, $\langle b_1, \cdots, b_g\rangle$ is mapped in the normal subgroup generated by $\langle b_1, \cdots, b_g\rangle$.

Now the induced automorphisms of α_1, $\beta_1\alpha_1^2\beta_1$, $\beta_i\alpha_i\gamma_i\beta_i$, $\beta_{i+1}\alpha_{i+1}\gamma_i\beta_{i+1}$ are given by

$$\alpha_1 \quad \begin{cases} a_1 \to b_1 a_1 \\ a_i \to a_i (i = 2, \cdots, g) \\ b_i \to b_i (i = 1, \cdots, g) \end{cases}$$

$$\beta_1\alpha_1^2\beta_1 \quad \begin{cases} a_1 \to a_1^{-1} b_1 a_1^{-1} b_1 a_1 \\ a_i \to a_i (i = 2, \cdots, g) \\ b_1 \to a_1^{-1} b_1^{-1} a_1 \\ b_i \to b_i (i = 2, \cdots, g), \end{cases}$$

$$\beta_i\alpha_i\gamma_i\beta_i \quad \begin{cases} a_i \to a_i^{-1} b_i a_i^{-1} b_i a_i b_{i+1}^{-1} \\ a_{i+1} \to b_{i+1} a_i^{-1} b_i^{-1} a_1^2 a_{i+1} \\ a_j \to a_j (j \neq i, i+1) \\ b_i \to b_{i+1} a_i^{-1} b_i^{-1} a_i \\ b_{i+1} \to b_{i+1} a_i^{-1} b_i^{-1} a_i^2 b_{i+1} a_i^{-2} b_i a_i b_{i+1}^{-1} \\ b_j \to b_j (j \neq i, i+1) \end{cases}$$

$\beta_{i+1}\alpha_{i+1}\gamma_i\beta_{i+1}$

$$\begin{cases} a_i \to b_i a_i b^{-1}{}_{i+1} a_{i+1} \\ a_{i+1} \to a_{i+1}^{-1} b_{i+1} a_i^{-1} b_i^{-1} a_i a_{i+1}^{-1} b_{i+1} a_{i+1} \\ a_j \to a_j (j \neq i, i+1) \\ b_{i+1} \to a_{i+1}^{-1} a_i^{-1} b_i a_i b_{i+1}^{-1} a_{i+1} \\ b_j \to b_j (j \neq i+1). \end{cases}$$

So, α_1, $\beta_1\alpha_1^2\beta_1$, $\beta_i\alpha_i\gamma_i\beta_i$, $\beta_{i+1}\alpha_{i+1}\gamma_i\beta_{i+1} \in \nu(\mathcal{M}_V)$.

Next we prove that $\alpha_i (i = 2, \cdots, g)$, $\gamma_i (i = 1, \cdots, g-1)$ and $\beta_i\alpha_i^2\beta_i (i = 2, \cdots, g)$ are generated by the elements stated in the theorem. Now,

$$\alpha_i(\beta_i\alpha_i\gamma_i\beta_i) = \beta_i\alpha_i\beta_i\gamma_i\beta_i = (\beta_i\alpha_i\gamma_i\beta_i)\gamma_i,$$
$$\begin{aligned} \gamma_i(\beta_{i+1}\alpha_{i+1}\gamma_i\beta_{i+1}) &= \gamma_i\beta_{i+1}\gamma_i\alpha_{i+1}\beta_{i+1} \\ &= \beta_{i+1}\gamma_i\beta_{i+1}\alpha_{i+1}\beta_{i+1} \\ &= \beta_{i+1}\gamma_i\alpha_{i+1}\beta_{i+1}\alpha_{i+1} \\ &= (\beta_{i+1}\alpha_{i+1}\gamma_i\beta_{i+1})\alpha_{i+1}. \end{aligned}$$

So,

$$\gamma_i = (\beta_i\alpha_i\gamma_i\beta_i)^{-1}\alpha_i(\beta_i\alpha_i\gamma_i\beta_i),$$
$$\alpha_{i+1} = (\beta_{i+1}\alpha_{i+1}\gamma_i\beta_{i+1})^{-1}\gamma_i(\beta_{i+1}\alpha_{i+1}\gamma_i\beta_{i+1}),$$

for $i = 1, \cdots, g-1$. Similarly, we have

$$\beta_{i+1}\alpha_{i+1}^2\beta_{i+1} = (\beta_i\alpha_i\gamma_i\beta_i)^{-1}(\beta_{i+1}\alpha_{i+1}\gamma_i\beta_{i+1})^{-1}(\beta_i\alpha_i\gamma_i\beta_i)$$
$$(\beta_i\alpha_i^2\beta_i)^{-1}(\beta_i\alpha_i\gamma_i\beta_i)(\beta_{i+1}\alpha_{i+1}\gamma_i\beta_{i+1})(\beta_i\alpha_i\gamma_i\beta_i).$$

These recursion formulae show that $\alpha_1 (i = 2, \cdots, g)$, $\gamma_i (i = 1, \cdots, g-1)$ and $\beta_1\alpha_i^2\beta_i (i = 2, \cdots, g)$ are generated by the elements stated in the theorem.

Finally we prove that τ_1, ω_1, θ_{12}, ξ_{12}, ρ_{12}, ρ are generated by the elements stated in the theorem. Now,

$$\tau_1 = \alpha_1^{-1},$$
$$\omega_1 = \alpha_1^2(\beta_1\alpha_1^2\beta_1),$$
$$\theta_{12} = \alpha_1^{-1}\alpha_2(\beta_2\alpha_2\gamma_1\beta_2)(\beta_2\alpha_2^2\beta_2)^{-1}\alpha_2^{-1},$$
$$\xi_{12} = \alpha_1^2(\beta_2\alpha_2^2\beta_2)\gamma_1^{-1}(\beta_2\alpha_2^2\beta_2)^{-1}\alpha_2^{-1}\alpha_1,$$
$$\rho_{12} = \alpha_1(\beta_1\alpha_1^2\beta_1)(\beta_1\alpha_1\gamma_1\beta_1)^{-1}(\beta_2\alpha_2\gamma_1\beta_2)^{-1}$$
$$\alpha_1^{-1}(\beta_1\alpha_1\gamma_1\beta_1)^{-1}(\beta_1\alpha_1^2\beta_1)\alpha_1.$$

It remains only to prove that ρ is generated by the elements stated in the theorem. Let

$$\rho_{ii+1} = \alpha_i(\beta_i\alpha_i^2\beta_i)(\beta_i\alpha_i\gamma_i\beta_i)^{-1}(\beta_{i+1}\alpha_{i+1}\gamma_i\beta_{i+1})^{-1}\alpha_i^{-1}$$
$$(\beta_i\alpha_i\gamma_i\beta_i)^{-1}(\beta_i\alpha_i^2\beta_i)\alpha_i.$$

The induced automorphism is given by

$$\begin{cases} a_i \to s_i^{-1} a_{i+1} s_i \\ a_{i+1} \to a_i \\ a_j \to a_j (j \neq i, i+1) \\ b_i \to s_i^{-1} b_{i+1} s_i \\ b_{i+1} \to b_i \\ b_j \to b_j (j \neq i, i+1). \end{cases}$$

Let $\theta = \rho_{g-1g}\rho_{g-2g-1}\cdots\rho_{23}\rho_{12}$. Then the induced automorphism is given by

$$\begin{cases} a_1 \to s_1^{-1} s_2^{-1} \cdots s_{g-1}^{-1} a_g s_{g-1} \cdots s_2 s_1 = s_g a_g s_g^{-1} \\ a_i \to a_{i-1} (i = 2, \cdots, g) \\ b_1 \to s_g b_g s_g^{-1} \\ b_i \to b_{i-1} (i = 2, \cdots, g). \end{cases}$$

Let $\eta = \theta\rho$. Then, the induced automorphism is given by

$$\begin{cases} a_1 \to s_1 a_1 s_1^{-1} \\ a_i \to a_i \quad (i = 2, \cdots, g) \\ b_1 \to s_1 b_1 s_1^{-1} \\ b_i \to b_i (i = 2, \cdots, g). \end{cases}$$

This means that $\eta = \alpha_1^2(\beta_1\alpha_1^2\beta_1)\alpha_1^2(\beta_1\alpha_1^2\beta_1)$. Hence ρ is generated by the elements stated in the theorem. This completes the proof of the theorem.

References

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