Small Stable Stationary Solutions in Morrey Spaces of the Navier-Stokes Equation

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Recently, many authors studied the Cauchy problem for the Navier-Stokes equation in $\boldsymbol{R}^n$ in the framework of Morrey spaces. For example, Giga and Miyakawa [2] and Kato [3] gave sufficient conditions for the unique existence of time-global solutions. For previous papers related to this problem, see the references of Kozono and Yamazaki [4], which studied the above Cauchy problem in new function spaces larger than the corresponding Morrey spaces. However, these papers considered only the case where the external force vanishes identically or decays as $t \to \infty$.

The purpose of this paper is to generalize the results on the global solvability in the works above to the case with a stationary external force by showing the unique existence and the stability of a small stationary solution in suitable Morrey spaces under appropriate assumptions on the external force.

More precisely, we consider the following stationary Navier-Stokes equation with an external force $f(x)$ in $\boldsymbol{R}^n$ for $n \geq 3$:

$$(1) \qquad -\Delta_x w(x) + (w(x) \cdot \nabla_x) w(x) + \nabla_x \pi(x) = f(x),$$

$$(2) \qquad \nabla_x \cdot w(x) = 0,$$

and find a sufficient condition on $f(x)$ for the unique existence of a small solution of (1)-(2) in suitable Morrey spaces.

We also verify the stability of the above stationary solution by showing the time-global unique solvability and giving a bound of the solution of the following nonstationary Navier-Stokes equation in $(0, \infty) \times \boldsymbol{R}^n_x$ with the same external force as above:

$$(3) \qquad \frac{\partial v}{\partial t}(t, x) - \Delta_x v(t, x) + (v(t, x) \cdot \nabla_x) v(t, x) + \nabla_x q(t, x) = f(x),$$

$$(4) \qquad \nabla_x \cdot v(t, x) = 0,$$

$$(5) \qquad v(0, x) = a(x) \text{ on } \boldsymbol{R}^n_x,$$

for the Cauchy data $a(x)$ close enough to the stationary solution.

Furthermore, we can take initial values in suitable function spaces introduced by [4]. These spaces are strictly larger than the corresponding Morrey spaces, and contain distributions other than Radon measures.

We start with the definition of the function spaces. Let p, q and s be real numbers such that $1 \leq q \leq p$, and suppose that $r \in [1, \infty]$. Then the *Morrey space* $\mathcal{M}_{p,q}$ on $\boldsymbol{R}^n$ is defined to be the set of functions $u(x) \in L^q_{\mathrm{loc}}(\boldsymbol{R}^n)$ such that

$$\| u \mid \mathcal{M}_{p,q} \| = \sup_{x_0 \in \boldsymbol{R}^n} \sup_{R>0} R^{n/p - n/q} \left(\int_{|x - x_0| < R} |u(x)|^q dx \right)^{1/q} < \infty.$$

We next define the space $\mathcal{M}^s_{p,q}$ by the formula

$$\mathcal{M}^s_{p,q} = \{ u(x) \in \mathcal{S}'/\mathcal{P} \mid \| u \mid \mathcal{M}^s_{p,q} \| = \| (-\Delta_x)^{s/2} u \mid \mathcal{M}_{p,q} \| < \infty \},$$

where $\mathcal{S}'$ and $\mathcal{P}$ denote the set of tempered distributions on $\boldsymbol{R}^n$ and the set of polynomials with n variables respectively.

Furthermore, we define the space $\mathcal{N}^s_{p,q,r}$ after [4] as the set of $u(x) \in \mathcal{S}'/\mathcal{P}$ such that

$$\| u \mid \mathcal{N}^s_{p,q,r} \| = \| \{ 2^{js} \| \mathcal{F}^{-1} [\varphi(2^{-j}\cdot) \mathcal{F}[u]] \mid \mathcal{M}_{p,q} \| \}^{\infty}_{j=-\infty} \mid \ell^r \| < \infty,$$

where $\{\varphi(2^{-j}\xi)\}^{\infty}_{j=-\infty}$ is a homogeneous Littlewood-Paley partition of unity. (See Bergh and Löfström [1] for example.)

Then it is shown in [4] that $\mathcal{N}^s_{p,q,1} \subset \mathcal{M}^s_{p,q} \subset \mathcal{N}^s_{p,q,\infty}$, and that the spaces $M^s_{p,q}$ and $\mathcal{N}^s_{p,q,r}$ can be canonically regarded as a subspace of $\mathcal{S}'$ if $s < n/p$.

Now we can state our main results.

Theorem A. *Suppose that r satisfies $2 < r \leq n$. Then there exist a positive number δ_0 and a continuous, strictly monotone-increasing function $\omega(\delta)$ on $[0, \delta_0]$ satisfying $\omega(0) = 0$ such that the following hold*:

(1) *For every $f(x) \in (\mathcal{D}')^n$, there exists at*

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most one solution $w(x)$ *of* (1)-(2) *in* $\mathcal{M}_{n,r}$ *satisfying the condition* $\| w \mid \mathcal{M}_{n,r} \| < \omega(\delta_0)$.

(2) *For every* $f(x) \in (\mathcal{M}_{n,r}^{-2})^n$ *satisfying* $\delta = \| f \mid \mathcal{M}_{n,r}^{-2} \| < \delta_0$, *there exists a solution* $w(x) \in (\mathcal{M}_{n,r})^n$ *of* (1)-(2) *such that* $\| u \mid \mathcal{M}_{n,r} \| \leq \omega(\delta)$.

Example 1. Suppose that $n \geq 4$, and let r be an integer satisfying $2 < r < n$. Next, let $\bar{w} = (w_1(x_1, \cdots, x_r), \cdots, w_r(x_1, \cdots, x_r))$ be a sufficiently small function in $L^r(\boldsymbol{R}^r)$ satisfying (2) on $\boldsymbol{R}^r$, and put $\bar{f} = -\Delta_x \bar{w} + \nabla_x(\bar{w} \otimes \bar{w})$ on $\boldsymbol{R}^r$. Then $w = (\bar{w}, 0)$ and $f = (\bar{f}, 0)$ satisfies (1)-(2) on $\boldsymbol{R}^n$. Moreover, we have $w \in \mathcal{M}_{n,r}$ and $f \in \mathcal{M}_{n,r}^{-2}$ on $\boldsymbol{R}^n$. Hence we can treat solutions $w(x)$ which does not necessarily satisfy $w(x) \to 0$ as $|x| \to \infty$.

Theorem B. *Let* r *be the same as in Theorem* A, *and suppose that* p, q *and* σ_0 *satisfy* $n/2 < p < \infty$, $1 < q \leq pr/n$ *and* $n/2p < \sigma_0 < \min\{1, n/p\}$. *Further, let* $w(x)$ *be the solution of* (1)-(2) *given in Theorem* A, (2). *Then there exists a positive number* $\delta_1 \leq \delta_0$ *such that, for every* $f(x) \in (\mathcal{M}_{n,r}^{-2})^n$ *satisfying* $\| f \mid \mathcal{M}_{n,r}^{-2} \| < \delta_1$, *there exist positive numbers* ε_0 *and* M_0 *such that, for every* $a(x) \in (\mathcal{M}_{p,q}^{n/p-1})^n$ *satisfying* $\nabla_x \cdot a(x) = 0$ *and* $\varepsilon = \| a(x) - w(x) \mid \mathcal{M}_{p,q}^{n/p-1} \| < \varepsilon_0$, *there uniquely exists a time-global solution* $v(t, x)$ *of* (3)-(4) *satisfying the conditions*

(6) $$\sup_{0<t\leq T'} t^{1/2-n/4p} \| v(t, \cdot) - w \mid \mathcal{M}_{p,q}^{n/2p} \| < \infty$$

for every $T' > 0$,

(7) $$\limsup_{t\to 0} t^{1/2-n/4p} \| v(t, \cdot) - w \mid \mathcal{M}_{p,q}^{n/2p} \| < M_0,$$

and the initial condition (5) *in the following sense*: *For every* s *such that* $-1 \leq s \leq n/p - 1$ *and for every* $T' > 0$, *we have*

(8) $$\sup_{0<t\leq T'} t^{s/2+1/2-n/2p} \| v(t, \cdot) - a \mid \mathcal{M}_{p,q}^{s} \| < \infty.$$

Moreover, for every T *such that* $0 < T \leq \infty$, *any solution of* (3)-(4) *on* $(0, T) \times \boldsymbol{R}^n$ *satisfying* (6) *for every* $T' \in (0, T)$, (7) *and* $v(t, \cdot) - a \to 0$ *in* $\mathcal{M}_{p,q}^{-1}$ *coincides with the restriction on* $(0, T) \times \boldsymbol{R}^n$ *of the above solution.*

Furthermore, for every σ *such that* $n/p - 1 \leq \sigma \leq \sigma_0$, *there exists a continuous, strictly monotone-increasing function* $\psi_\sigma(\varepsilon)$ *on* $[0, \varepsilon_0]$ *satisfying* $\psi_\sigma(0) = 0$ *such that the estimate*

(9) $$\sup_{t>0} t^{\sigma/2+1/2-n/2p} \| v(t, \cdot) - w \mid \mathcal{M}_{p,q}^{\sigma} \| \leq \psi_\sigma(\varepsilon)$$

holds if $\varepsilon < \varepsilon_0$.

Remark 2. If $p \geq n$, we have $\mathcal{M}_{n,r} \subset \mathcal{M}_{p,pr/n}^{n/p-1} \subset \mathcal{M}_{p,q}^{n/p-1}$. Hence the estimate (9) with $\sigma = n/p - 1$ in Theorem B, together with the fact $\lim_{\varepsilon\to+0} \psi_{n/p-1}(\varepsilon) = 0$, asserts the Lyapunov stability of the stationary solution $w(x)$ in the topology of $\mathcal{M}_{p,q}^{n/p-1}$. In particular, we can put $p = n$ and $q = r$ in Theorem B, and in this case the above fact implies the Lyapunov stability of $w(x)$ in the space $\mathcal{M}_{n,r}$ itself.

The estimate (9) with $\sigma > n/p - 1$ asserts the asymptotic stability of $w(x)$ in different topologies; more precisely, the rate of the convergence in $\mathcal{M}_{p,q}^{\sigma}$.

Remark 3. In Theorem B, the solution $v(t, x)$ may not be strongly continuous at $t = 0$ in $\mathcal{M}_{p,q}^{n/p-1}$. Hence, in order to ensure the uniqueness, we must assume a condition on some sort of smallness near $t = 0$ like (7).

Example 4. Suppose that $1 \leq n_1 < n$, and put $x' = (x_1, \cdots, x_{n_1})$ for $x = (x_1, \cdots, x_n)$. Then we have

$$(-\Delta_x)^{(n-n_1p)/2p} \delta(x') = c\, |x'|^{-n/p} \in \mathcal{M}_{p,q}$$

on $\boldsymbol{R}^n$ for every p and q such that $1 < q < n_1 p/n$.

In particular, we can take $a(x) = w(x) + \varepsilon\,(0, \cdots, 0, \delta(x_1))$ in Theorem B provided the constant ε is sufficiently small, since $\delta(x_1) \in \mathcal{M}_{p,q}^{n/p-1}$ holds for every p and q such that $1 < q < p/n$. This case is treated by Kato [3].

Next, in view of the Biot-Savard law, we can take

$$a(x) = w(x) + \varepsilon \left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0, \cdots, 0 \right)$$

in Theorem B provided the constant ε is sufficiently small, since

$$(-\Delta_x)^{-1/2} \delta(x_1, x_2) \in \mathcal{M}_{p,q}^{n/p-1}$$

holds for every p and q such that $1 < q < 2p/n$. This case is treated by Giga and Miyakawa [2].

Theorem C. *Let* p, q, r, σ_0, $f(x)$ *and* $w(x)$ *be the same as in Theorem* B. *Then there exist positive numbers* ε_1 *and* M_1 *such that, for every* $a(x) \in (\mathcal{N}_{p,q,\infty}^{n/p-1})^n$ *satisfying* $\nabla_x \cdot a(x) = 0$ *and* $\varepsilon = \| a(x) - w(x) \mid \mathcal{N}_{p,q,\infty}^{n/p-1} \| < \varepsilon_1$, *there uniquely exists a time-global solution* $v(t, x)$ *of* (3)-(4) *satisfying the conditions* (6) *for every* $T' \in (0, \infty)$, (7) *with* M_0 *replaced by* M_1, *and* (8) *for every* s *such that* $-1 \leq s < n/p - 1$ *and every* $T' > 0$.

Moreover, for every T *such that* $0 < T \leq \infty$, *any solution of* (3)-(4) *on* $(0, T) \times \boldsymbol{R}^n$ *satisfying*

(6) *for every* $T' \in (0, T)$, (7) *and* $v(t, \cdot) - a \to 0$ *in* $\mathcal{M}_{p,q}^{-1}$ *coincides with the restriction on* $(0, T) \times \boldsymbol{R}^n$ *of the above solution.*

Furthermore, there exists a continuous, strictly monotone-increasing function $\bar{\psi}(\varepsilon)$ *on* $[0, \varepsilon_1]$ *satisfying* $\bar{\psi}(0) = 0$ *such that the estimate*

(10) $$\sup_{t>0} \| v(t, \cdot) - w \mid \mathcal{N}_{p,q,\infty}^{n/p-1} \| \leq \bar{\psi}(\varepsilon)$$

holds if $\varepsilon < \varepsilon_1$, *and for every* σ *such that* $n/p - 1 < \sigma < \sigma_0$, *there exists a continuous, strictly monotone-increasing function* $\psi'_\sigma(\varepsilon)$ *on* $[0, \varepsilon_1]$ *satisfying* $\psi'_\sigma(0) = 0$ *such that the estimate*

(11) $$\sup_{t>0} t^{\sigma/2+1/2-n/2p} \| v(t, \cdot) - w \mid \mathcal{M}_{p,q}^{\sigma} \| \leq \psi'_\sigma(\varepsilon)$$

holds if $\varepsilon < \varepsilon_1$.

Example 5. It was shown in [4] that the distribution p.v. $\dfrac{1}{x_1}$ belongs to the space $\mathcal{N}_{p,p/n,\infty}^{n/p-1}$ if $p > n$, and $q = p/n$ enjoys the condition of Theorem C for every r. Hence we can take $a = w + \varepsilon \left(0, \cdots, 0, \text{p.v.}\dfrac{1}{x_1}\right)$ in Theorem C, provided the constant ε is sufficiently small.

Remark 6. If $p \geq n$, Remark 2 and the property of the space $\mathcal{N}_{p,q,r}^{s}$ imply the inclusion relation $\mathcal{M}_{n,r} \subset \mathcal{M}_{p,q}^{n/p-1} \subset \mathcal{N}_{p,q,\infty}^{n/p-1}$. Hence the estimate (10) and the fact $\lim_{\varepsilon \to +0} \bar{\psi}(\varepsilon) = 0$ assert the Lyapunov stability of the stationary solution $w(x)$ in the topology of $\mathcal{N}_{p,q,\infty}^{n/p-1}$.

The estimate (11) asserts the asymptotic stability of $w(x)$ in different topologies; more precisely, the rate of the convergence in $\mathcal{M}_{p,q}^{\sigma}$.

Remark 7. The main result of [4] on the Navier-Stokes equation can be regarded as the stability of the stationary solution 0 of the equation (1)-(2) with the external force $f \equiv 0$. Hence Theorem 3 can be regarded as a generalization of the above results to the case with more general stationary solutions. As is described in these papers, our function spaces is strictly larger than the ones considered in [2] and [3].

Details will be published elsewhere.

References

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