Quadratic Relations between Logarithms of Algebraic Numbers

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So far, the four exponentials conjecture has been solved only in one special case, namely when the transcendence degree of the field which is spanned by the four logarithms is 1. We produce a new proof of this statement, and we announce a generalization: we replace the determinant of a 2×2 matrix by any homogeneous polynomial of degree 2.

§1. The results. The following statement provides a solution of the four exponentials conjecture in transcendence degree 1.

Theorem 1. *Let x_1 and x_2 be two complex numbers which are linearly independent over $\boldsymbol{Q}$, and similarly let y_1, y_2 be two $\boldsymbol{Q}$-linearly independent complex numbers. Assume that the field $\boldsymbol{Q}(x_1, x_2, y_1, y_2)$ has transcendence degree 1 over $\boldsymbol{Q}$. Then one at least of the four numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_2y_1},\ e^{x_2y_2}$$

is transcendental.

For a proof of this result, we refer to [1] Cor. 7 and [6] Cor. 4.

Our is goal to introduce a sketch of new proof, where Gel'fond's transcendence criterion [3] (Chap. III, §4, lemma VII) is replaced by a diophantine approximation result due to Wirsing [8] §3. This new argument allows us to use Laurent's interpolations determinants [4] §6.

A generalization of theorem 1 can be achieved with the same arguments. Here we merely state the result; a complete proof (of a more general statement) will be given in another paper.

We denote by $\mathscr{L}$ the $\boldsymbol{Q}$-vector space of logarithms of algebraic numbers:

$$\mathscr{L} = \exp^{-1}(\bar{\boldsymbol{Q}}^{\times}) = \{z \in \boldsymbol{C}\,;\, e^z \in \bar{\boldsymbol{Q}}^{\times}\} \subset \boldsymbol{C},$$

where $\bar{\boldsymbol{Q}}$ is the algebraic closure of $\boldsymbol{Q}$ in $\mathbf{C}$ and $\bar{\boldsymbol{Q}}^{\times}$ is the multiplicative group of non-zero algebraic numbers.

Theorem 2. *Let $V \subseteq \mathbf{C}^n$ be the set of zeroes in $\boldsymbol{C}^n$ of a non-zero homogeneous polynomial $P \in \boldsymbol{Q}[X_1, \dots, X_n]$ of degree ≤ 2 and let $(\lambda_1, \dots, \lambda_n)$ be a point in V with coordinates in $\mathscr{L}$. Assume that the field $\boldsymbol{Q}(\lambda_1, \dots, \lambda_n)$ has transcendence degree 1 over $\boldsymbol{Q}$. Then $(\lambda_1, \dots, \lambda_n)$ is contained in a vector subspace of $\boldsymbol{C}^n$ which is defined over $\boldsymbol{Q}$ and contained in V.*

Theorem 1 is the special case of Theorem 2 when P is $X_1X_4 - X_2X_3$ with $n = 4$.

§2. Wirsing's theorem. When α is a complex algebraic number of degree $d = [\boldsymbol{Q}(\alpha) : \boldsymbol{Q}]$, we denote by $\mathrm{M}(\alpha)$ its Mahler's measure, which is related to its absolute logarithmic height $\mathrm{h}(\alpha)$ by

$$d\mathrm{h}(\alpha) = \log \mathrm{M}(\alpha).$$

The main tool of this paper is the following theorem of Wirsing [8]:

Theorem 3. *Let θ be a complex transcendental number. For any integer $D \geq 2$ there exist infinitely many algebraic numbers α which satisfy*

$$[\boldsymbol{Q}(\alpha) : \boldsymbol{Q}] \leq D \text{ and } |\theta - \alpha| \leq \mathrm{M}(\alpha)^{-D/4}.$$

§3. Laurent's interpolation determinants. A proof of the six exponentials theorem which does not rest on Dirichlet's box principle has been given by M. Laurent in [4]: he replaces the construction of an auxiliary function (which involves Thue-Siegel lemma) by an explicit determinant.

The following result is a variant of Proposition 9.5 of [7]: here, we include derivatives.

Proposition. *Let L be a positive integer, E and U be positive real numbers with $0 < \log E \leq 4U$. For $1 \leq \lambda \leq L$, let φ_λ be a complex function of one variable, $b_{\lambda 1}, \dots, b_{\lambda L}$ be complex numbers and M_λ a real number; further, for $1 \leq \mu \leq L$, let ζ_μ be a complex number and σ_μ be a non-negative integer. Assume that, for $1 \leq \lambda \leq L$, we have*

$$M_\lambda \geq \log \sup_{|z|=E} \max_{1 \leq \mu \leq L} |((d/dz)^{\sigma_\mu}\varphi_\lambda)(z\zeta_\mu)|$$
$$\text{and } M_\lambda \geq \log \max_{1 \leq \mu \leq L} |b_{\lambda\mu}|.$$

Finally, let ε be a complex number with $|\varepsilon| \leq e^{-U}$. Then the logarithm of the absolute value of the determinant

$$\Delta = \det((d/dz)^{\sigma_\mu}\varphi_\lambda(\zeta_\mu) + \varepsilon b_{\lambda\mu})_{1 \leq \lambda,\mu \leq L}$$

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is bounded by

$$\log|\Delta| \leq -LU + \frac{U^2}{2\log E} + U + L\log(2L) + \sum_{\mu=1}^{L}\sigma_\mu \log E + \sum_{\lambda=1}^{L} M_\lambda.$$

§4. Philippon's zero estimate. Let Γ be a finitely generated subgroup of $\boldsymbol{C} \times (\boldsymbol{C}^\times)^2$. We fix a finite set of generators $\gamma_1, \dots, \gamma_l$ of Γ, and for any positive integer S we define a subset $\Gamma(S)$ of Γ by

$$\Gamma(S) = \{s_1\gamma_1 + \cdots + s_l\gamma_l\,;\,(s_1, \dots, s_l) \in \mathbf{Z}^l,\ 0 \leq s_j \leq S,\ (1 \leq j \leq l)\}.$$

We denote by Γ_1 the projection of Γ on $(\boldsymbol{C}^\times)^2$ and by $\Gamma_1(S)$ the same projection of $\Gamma(S)$. With these notations, we will need the following special case of Philippon's zero estimate [5].

Lemma. *Let a and b be two complex numbers and $\mathfrak{D}$ the derivative operator*

$$(\partial/\partial X_0) + aX_1(\partial/\partial X_1) + bX_2(\partial/\partial X_2).$$

Let T_0, T_1, S_0 and S_1 be positive integers which satisfy

$$S_1 > 18T_1 \text{ and } S_0 > 18T_0.$$

Assume that there exists a non-zero polynomial $P \in \boldsymbol{C}[X_0, X_1, X_2]$, of degree $\leq T_0$ with respect to X_0, of degree $\leq T_1$ with respect to each of the variables X_1 and X_2 such that

$$\mathfrak{D}^\sigma P(\gamma) = 0 \text{ for all } \gamma \in \Gamma(S_1) \text{ and } 0 \leq \sigma \leq S_0.$$

Then either Γ_1 is of rank ≤ 1, or else there exists an algebraic subgroup G_1 of $\boldsymbol{G}_m^2$, of dimension 1, such that $\Gamma_1 \cap G_1$ is of finite index in Γ_1.

Proof. Consider the algebraic group $G = \boldsymbol{G}_a \times \boldsymbol{G}_m^2$ and identify its tangent space at the origin with $\boldsymbol{C}^3$ so that the exponential map $\exp_G$ of the complex Lie group $G(\boldsymbol{C}) = \boldsymbol{C} \times (\boldsymbol{C}^\times)^2$ be given by $\exp_G(z_0\,;\,z_1, z_2) = (z_0\,;\,e^{z_1}, e^{z_2}) \in G(\boldsymbol{C})$. The assumption is that the polynomial P vanishes at each point of $\Gamma(S_1)$ with a multiplicity $\geq S_0 + 1$ along the one-parameter analytic subgroup $A = \exp_G(W)$ where $W = \boldsymbol{C}(1\,;\,a, b) \subset \boldsymbol{C}^3$. By virtue of théorème 2.1 of [5] (which, in this case, holds with $c = 1$), there exists a connected algebraic subgroup G' of G with $G' \neq G$ such that, if we write $G' = \boldsymbol{G}_a^{\delta_0} \times G_1$ with $G_1 \subset \boldsymbol{G}_m^2$, then we have

$$[S_0/3]^{1-\delta_0}[S_1/3]^{l'}T_0^{\delta_0}T_1^{\delta_1} \leq 3!\,T_0T_1^2$$

where $\delta_1 = \dim(G_1)$, $\operatorname{rank}(\Gamma_1/(\Gamma_1 \cap G_1(\boldsymbol{C})))$ and the brackets $[\cdot]$ denote the integral part. Since $S_i > 18T_i$ for $i = 1,2$, we deduce $\delta_1 < 2$ and $l' < 2 - \delta_1$. If $\delta_1 = 0$, this gives $\operatorname{rank}(\Gamma_1) < 2$. If $\delta_1 = 1$, we get $\operatorname{rank}(\Gamma_1/(\Gamma_1 \cap G_1(\boldsymbol{C}))) = 0$.

§5. Sketch of proof of theorem 1. Let x_1, x_2, y_1, y_2 be four complex numbers in a field of transcendence degree 1 over $\boldsymbol{Q}$ such that the four numbers $\alpha_{ij} = e^{x_iy_j}$, $(i = 1,2,\ j = 1,2)$ are algebraic. For simplicity we assume that there exists a complex number θ such that

$x_i = A_i(\theta)$, $(i = 1,2)$ and $y_j = B_j(\theta)$, $(j = 1,2)$, where A_1, A_2, B_1, B_2 are four polynomials in $\bar{\boldsymbol{Q}}[X]$.

Our proof involves analytic functions of one variable: for non-negative integers τ, t_1, t_2, define

$$\varphi_{\tau t}(z) = z^\tau e^{(t_1x_1+t_2x_2)z},$$

where the subscript t stands for (t_1, t_2). Further, when σ, s_1, s_2 are non-negative integers, we define polynomials $P_{\tau t}^{\sigma s}$ in $\bar{\boldsymbol{Q}}[X]$ by

$$P_{\tau t}^{\sigma s} = \sum_{\kappa=0}^{\min\{\sigma,\tau\}} \frac{\sigma!\,\tau!}{\kappa!\,(\sigma-\kappa)!\,(\tau-\kappa)!} \times (t_1A_1 + t_2A_2)^{\sigma-\kappa}(s_1B_1 + s_2B_2)^{\tau-\kappa}\alpha_{11}^{t_1s_1}\alpha_{12}^{t_1s_2}\alpha_{21}^{t_2s_1}\alpha_{22}^{t_2s_2}.$$

According to this definition, it is readily checked that we have

$$P_{\tau t}^{\sigma s}(\theta) = \left(\frac{d}{dz}\right)^\sigma \varphi_{\tau t}(s_1y_1 + s_2y_2).$$

The numbers c_1, c_2, c_3, c_4, c_5 which appear below denote effectively computable constants, which depend only on the absolute value of θ and the heights and degrees of the algebraic numbers α_{ij} as well as the heights and degrees of the coefficients of the polynomials A_i and B_j. We let $c > 0$ denote an integer which is say 16 times larger than the maximum of the c_i's. Next, we choose a positive integer D, which is bounded from below by some function of c which could be made explicit. Finally we choose a positive integer N_0, which is large compared wlth D, and we use Wirsing's theorem 3: there exists an algebraic number α which satisfies

$$[\boldsymbol{Q}(\alpha) : \boldsymbol{Q}] \leq D,\ N = \log \mathrm{M}(\alpha) \geq N_0 \text{ and } |\theta - \alpha| \leq e^{-DN/4}.$$

We define positive integers T_0, T_1, S_0, S_1 by the following conditions:

$T_0 = [c^{-2}D]$, $T_1 = [c^{-1}N^{1/2}]$, $S_0 = cT_0$, $S_1 = cT_1$.

We set $L = (T_0 + 1)(T_1 + 1)^2$ and we choose any ordering of the set of L triples (τ, t_1, t_2) which satisfy

$$0 \leq \tau \leq T_0,\ 0 \leq t_1 \leq T_1,\ 0 \leq t_2 \leq T_1.$$

Similarly we choose any ordering of the set of triples (σ, s_1, s_2) which satisfy

$$0 \le \sigma \le S_0,\ 0 \le s_1 \le S_1,\ 0 \le s_2 \le S_1.$$

This enables us to build a matrix $\mathcal{M}(X) = (P_{\tau t}^{\sigma s}(X))$ with coefficients in $\bar{\boldsymbol{Q}}[X]$. We wish to prove that the specialization $\mathcal{M}(\alpha)$ of this matrix at the point α has rank $< L$. In order to achieve this goal, we consider an arbitrary subset of triples (σ, s_1, s_2) with L elements, and we introduce the determinant $\Delta(X)$ of the corresponding $L \times L$ matrix. We claim that $\Delta(\alpha)$ vanishes.

We use the proposition of §3 for the functions $\varphi_{\tau t}$ with

$$E = D,\ U = DN/4 \text{ and } \varepsilon = e^{-U}.$$

For all $(\tau, t, \sigma, s) \in [0, T_0] \times [0, T_1]^2 \times [0, S_0] \times [0, S_1]^2$, we define a complex number $b_{\tau t}^{\sigma s}$ by

$$\varepsilon b_{\tau t}^{\sigma s} = P_{\tau t}^{\sigma s}(\alpha) - P_{\tau t}^{\sigma s}(\theta),$$

and we observe that the estimates

$$\sup_{|z|=E} \log |((d/dz)^{\sigma_\mu}\varphi_{\tau t})(z(s_1 y_1 + s_2 y_2))| \le M$$

$$\text{and } \log |b_{\tau t}^{\sigma s}| \le M$$

hold with

$$M = c_1(T_0 \log(T_0 S_1 E) + S_0 \log(T_1 S_0) + T_1 S_1 E).$$

Since

$$\frac{U^2}{2L\log E} + \frac{U}{L} + \log(2L) + S_0 \log E + M < \frac{1}{2}U,$$

we conclude

$$|\Delta(\alpha)| \le e^{-LDN/8}.$$

The number $\Delta(\alpha)$ is the value of a polynomial Q, with rational integer coefficients, at an algebraic point; the coordinates of this algebraic point are $\alpha, \alpha_{11}, \alpha_{21}, \alpha_{12}, \alpha_{22}$, as well as the coefficients of the polynomials A_1, A_2, B_1, B_2. The length of Q is at most

$$L!(c_2 T_1 S_0)^{LS_0}(c_2 T_0 S_1)^{LT_0}.$$

The degree of Q with respect to the variable corresponding to α is at most $c_3 L(T_0 + S_0)$; for $i = 1,2$ and $j = 1,2$, the degree with respect to the variable corresponding to α_{ij} is at most LT_1S_1 ; finally the degree with respect to each of the other variables is at most $L \max\{T_0, S_0\}$. From Liouville's inequality (see for instance [2] Lemma 9.2) we deduce that either $\Delta(\alpha) \ne 0$ or else

$$\frac{1}{L}\log|\Delta(\alpha)| \ge -c_4 D(T_0 \log(T_0 S_1) + S_0 \log(T_1 S_0) + T_1 S_1) - c_4(T_0 + S_0)\log \mathrm{M}(\alpha) \ge -c_5 c^{-1} DN.$$

According to the previous upper bound for $|\Delta(\alpha)|$, we deduce $\Delta(\alpha) = 0$.

This completes the proof of our claim: the matrix $\mathcal{M}(\alpha)$ has rank $< L$. Therefore, if we set, for $s = (s_1, s_2) \in \mathbf{Z}^2$,

$$\gamma_s = (s_1 B_1(\alpha) + s_2 B_2(\alpha)\,;\alpha_{11}^{s_1}\alpha_{12}^{s_2}, \alpha_{21}^{s_1}\alpha_{22}^{s_2}) \in \boldsymbol{C} \times (\boldsymbol{C}^\times)^2,$$

and if $\mathcal{D}$ denotes the derivative operator

$$(\partial/\partial X_0) + A_1(\alpha)X_1(\partial/\partial X_1) + A_2(\alpha)X_2(\partial/\partial X_2),$$

then there exist complex numbers $p_{\tau t}$, not all of which are zero, such that the polynomial

$$P(X_0, X_1, X_2) = \sum_{\tau=0}^{T_0}\sum_{t_1=0}^{T_1}\sum_{t_2=0}^{T_1} p_{\tau t} X_0^\tau X_1^{t_1} X_2^{t_2} \in \boldsymbol{C}[X_0, X_1, X_2]$$

satisfies

$$\mathcal{D}^\sigma P(\gamma_s) = 0 \text{ for } 0 \le \sigma \le S_0,\ 0 \le s_i \le S_1,\ (i = 1,2).$$

We define Γ_1 as the subgroup of $(\boldsymbol{C}^\times)^2$ which is generated by $(\alpha_{11}, \alpha_{21})$ and $(\alpha_{12}, \alpha_{22})$. From the zero estimate (lemma in section 4), we deduce that either Γ_1 is of rank ≤ 1, or else there exists an algebraic subgroup G_1 of $\boldsymbol{G}_m^2$ of dimension 1 such that $\Gamma_1 \cap G_1(\boldsymbol{C})$ is of finite index in Γ_1. It follows that either x_1 and x_2 are linearly dependent over $\boldsymbol{Q}$, or else y_1 and y_2 are linearly dependent over $\boldsymbol{Q}$. This completes the sketch of proof of Theorem 1.

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