A Construction of Exceptional Simple Graded Lie Algebras of the Second Kind

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(Communicated by Kiyosi ITÔ, M. J. A., Jan. 12, 1995)

§0. Introduction. Let $\mathfrak{g} = \sum_{k=-2}^{2} g_k$ be a *graded Lie algebra of the second kind* (shortly 2-GLA). In [5], Kaneyuki gave the classification of exceptional real simple 2-GLA's and listed up the subalgebras g_0 and the dimension of $g_k (k = 1, 2)$. Since the subspaces $g_k (k \neq 0)$ were not explicitly determined in [5], we will give an explicit representation of g_k in this paper. Up to the present, several constructions of 2-GLA have been thought out. Allison ([1]) gave a construction of 2-GLA starting from structurable algebra. His construction is useful but some exceptional real simple 2-GLA's can not be obtained by his construction. Details and proofs will be found in [3].

§1. Methods of construction. In this section, we give two methods of construction of 2-GLA.

1.1 Let g_0 be a real Lie algebra and $V_k (k = 1, 2)$ a real vector space with a nondegenerate symmetric bilinear form (,). For each element $\boldsymbol{u}$ of V_k, the element $\boldsymbol{u}^*$ of the dual space V_k^* is defined by $\boldsymbol{u}^*(\boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v})$ $(\boldsymbol{v} \in V_k)$. Let ρ_k be a representation of g_0 on $V_k (k = 1, 2)$. By ρ_k^*, we denote the dual representation of ρ_k, that is

$$(\rho_k^*(X)\boldsymbol{u}^*)(\boldsymbol{v}) + \boldsymbol{u}^*(\rho_k(X)\boldsymbol{v}) = 0 \quad (\boldsymbol{u}, \boldsymbol{v} \in V_k, X \in g_0).$$

Now, we assume that the following bilinear maps are given.

$$\Delta : V_2 \times V_1^* \to V_1, \quad \circ : V_1 \times V_1 \to V_2 \ \text{(antisymmetric)},$$
$$\times : V_1 \times V_1^* \to g_0, \quad * : V_2 \times V_2^* \to g_0.$$

Let us consider the real vector space

$$\mathfrak{g} = g_0 \oplus V_1 \oplus V_1^* \oplus V_2 \oplus V_2^*.$$

We define a bilinear bracket operation in $\mathfrak{g}$ as follows:

$$(X, \boldsymbol{u}, \boldsymbol{v}^*, \boldsymbol{x}, \boldsymbol{y}^*) = [(X_1, \boldsymbol{u}_1, \boldsymbol{v}_1^*, \boldsymbol{x}_1, \boldsymbol{y}_1^*), (X_2, \boldsymbol{u}_2, \boldsymbol{v}_2^*, \boldsymbol{x}_2, \boldsymbol{y}_2^*)],$$

where

$$\begin{cases} X = [X_1, X_2] + \boldsymbol{u}_1 \times \boldsymbol{v}_2^* - \boldsymbol{u}_2 \times \boldsymbol{v}_1^* + \boldsymbol{x}_1 * \boldsymbol{y}_2^* - \boldsymbol{x}_2 * \boldsymbol{y}_1^*, \\ \boldsymbol{u} = \rho_1(X_1)\boldsymbol{u}_2 - \rho_1(X_2)\boldsymbol{u}_1 + \boldsymbol{x}_1 \Delta \boldsymbol{v}_2^* - \boldsymbol{x}_2 \Delta \boldsymbol{v}_1^*, \\ \boldsymbol{v}^* = \rho_1^*(X_1)\boldsymbol{v}_2^* - \rho_1^*(X_2)\boldsymbol{v}_1^* - (\boldsymbol{y}_1 \Delta \boldsymbol{u}_2^*)^* + (\boldsymbol{y}_2 \Delta \boldsymbol{u}_1^*)^*, \\ \boldsymbol{x} = \rho_2(X_1)\boldsymbol{x}_2 - \rho_2(X_2)\boldsymbol{x}_1 + \boldsymbol{u}_1 \circ \boldsymbol{u}_2, \\ \boldsymbol{y}^* = \rho_2^*(X_1)\boldsymbol{y}_2^* - \rho_2^*(X_2)\boldsymbol{y}_1^* - (\boldsymbol{v}_1 \circ \boldsymbol{v}_2)^*. \end{cases}$$

In [3], we give a necessary and sufficient condition for $\mathfrak{g}$ to be a Lie algebra. When $\mathfrak{g}$ is a Lie algebra, obviously $\mathfrak{g} = \sum_{k=-2}^{2} g_k (g_k = V_k, g_{-k} = V_k^*)$ becomes a 2-GLA.

1.2. Let $\mathfrak{g} = \sum_{k=-2}^{2} g_k$ be a 2-GLA and γ a grade-preserving involution (= involutive automorphism) of $\mathfrak{g}$. Put

$$\mathfrak{g}_\gamma = \{X \in \mathfrak{g} \mid \gamma(X) = X\},$$
$$(g_k)_\gamma = \{X \in g_k \mid \gamma(X) = X\}.$$

If $(g_{\pm 2})_\gamma \neq (0)$, then the subalgebra $\mathfrak{g}_\gamma = \sum_{k=-2}^{2} (g_k)_\gamma$ also becomes a 2-GLA.

§2. The main theorem. Using g_0 and dim g_k listed up in [5], we construct the corresponding 2-GLA's by the methods described in §1. Then we have the following theorem.

Theorem 1. *The exceptional real simple graded Lie algebras of the second kind are realized as listed in Table* I.

In Table I, we use the following notations.

$\boldsymbol{C}$ (resp. $\boldsymbol{C}'$): the algebra of complex (resp. split complex) numbers

$\boldsymbol{H}$ (resp. $\boldsymbol{H}'$): the algebra of quaternion (resp. split quaternion) numbers

$\mathfrak{C}$ (resp. $\mathfrak{C}'$): the division Cayley (resp. split Cayley) algebra

For a real vector space V, its complexification $\{\boldsymbol{u} + i\boldsymbol{v} \mid \boldsymbol{u}, \boldsymbol{v} \in V\}$ is denoted by V^C. We do not identify $\boldsymbol{R}^C$ with $\boldsymbol{C}$, but denote $\boldsymbol{R}^C$ by C.

From now on, we explain the contents of Table I.

(1) In case of (e1)~(e9) and (e24)~(e27):

Table I

	$\mathfrak{g}$	g_0	g_1	g_2
(e1)	$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_{7(7)} \oplus \boldsymbol{R}$	$\mathfrak{P}_{\mathfrak{C}'}$	$\boldsymbol{R}$
(e2)	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_{7(-25)} \oplus \boldsymbol{R}$	$\mathfrak{P}_{\mathfrak{C}}$	$\boldsymbol{R}$
(e3)	$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(3, \boldsymbol{R}) \oplus \boldsymbol{R}$	$\mathfrak{P}_{\boldsymbol{R}}$	$\boldsymbol{R}$
(e4)	$\mathfrak{e}_{6(6)}$	$\mathfrak{sl}(6, \boldsymbol{R}) \oplus \boldsymbol{R}$	$\mathfrak{P}_{\boldsymbol{C}'}$	$\boldsymbol{R}$
(e5)	$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(3,3) \oplus \boldsymbol{R}$	$\mathfrak{P}_{\boldsymbol{C}}$	$\boldsymbol{R}$
(e6)	$\mathfrak{e}_{7(7)}$	$\mathfrak{so}(6,6) \oplus \boldsymbol{R}$	$\mathfrak{P}_{\boldsymbol{H}'}$	$\boldsymbol{R}$
(e7)	$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}^*(12) \oplus \boldsymbol{R}$	$\mathfrak{P}_{\boldsymbol{H}}$	$\boldsymbol{R}$
(e8)	$\mathfrak{e}_{6(-14)}$	$\mathfrak{su}(1,5) \oplus \boldsymbol{R}$	$(\mathfrak{P}^C_{\boldsymbol{C}'})_{+\tau}$	$i\boldsymbol{R}$
(e9)	$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}(2,10) \oplus \boldsymbol{R}$	$(\mathfrak{P}^C_{\boldsymbol{H}'})_{+\tau}$	$i\boldsymbol{R}$
(e10)	$\mathfrak{g}_{2(2)}$	$\mathfrak{sl}(2, \boldsymbol{R}) \oplus \boldsymbol{R}$	$\boldsymbol{R}_3[e_1, e_2]$	$\boldsymbol{R}$
(e11)	$\mathfrak{e}_{6(6)}$	$\mathfrak{so}(4,4) \oplus \boldsymbol{R} \oplus \boldsymbol{R}$	$\mathfrak{C}' \oplus \mathfrak{C}'$	$\mathfrak{C}'$
(e12)	$\mathfrak{e}_{6(-26)}$	$\mathfrak{so}(8) \oplus \boldsymbol{R} \oplus \boldsymbol{R}$	$\mathfrak{C} \oplus \mathfrak{C}$	$\mathfrak{C}$
(e13)	$\mathfrak{e}_{6(2)}$	$\mathfrak{so}(3,5) \oplus \boldsymbol{R} \oplus i\boldsymbol{R}$	$\mathfrak{C}' \oplus i\mathfrak{C}'$	$\mathfrak{C}'_{00}$
(e14)	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(1,7) \oplus \boldsymbol{R} \oplus i\boldsymbol{R}$	$\mathfrak{C} \oplus i\mathfrak{C}$	$\mathfrak{C}_{00}$
(e15)	$\mathfrak{f}_{4(4)}$	$\mathfrak{so}(3,4) \oplus \boldsymbol{R}$	$\mathfrak{C}'$	$\mathfrak{C}'_0$
(e16)	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(7) \oplus \boldsymbol{R}$	$\mathfrak{C}$	$\mathfrak{C}_0$
(e17)	$\mathfrak{e}_{6(6)}$	$\mathfrak{sl}(5, \boldsymbol{R}) \oplus \mathfrak{sl}(2, \boldsymbol{R}) \oplus \boldsymbol{R}$	$\boldsymbol{R}^2 \oplus (\boldsymbol{R}^5)_2$	$(\boldsymbol{R}^5)_4$
(e18)	$\mathfrak{e}_{7(7)}$	$\mathfrak{sl}(7, \boldsymbol{R}) \oplus \boldsymbol{R}$	$(\boldsymbol{R}^7)_3$	$(\boldsymbol{R}^7)_6$
(e19)	$\mathfrak{e}_{7(7)}$	$\mathfrak{so}(5,5) \oplus \mathfrak{sl}(2, \boldsymbol{R}) \oplus \boldsymbol{R}$	$\boldsymbol{R}^2 \oplus \wedge^+(U_{(5)})$	$W_{(5)}$
(e20)	$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}(1,9) \oplus \mathfrak{sl}(2, \boldsymbol{R}) \oplus \boldsymbol{R}$	$\boldsymbol{R}^2 \oplus (\wedge^+(U_{(5)})^C)_{l_{1,9}}$	$(W^C_{(5)})_{l_{1,9}}$
(e21)	$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(3,7) \oplus \mathfrak{su}(2) \oplus \boldsymbol{R}$	$(\boldsymbol{R}^2 \oplus \wedge^+(U_{(5)})^C)_{+l_{3,7}}$	$(W^C_{(5)})_{l_{3,7}}$
(e22)	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(7,7) \oplus \boldsymbol{R}$	$\wedge^+(U_{(7)})$	$W_{(7)}$
(e23)	$\mathfrak{e}_{8(-24)}$	$\mathfrak{so}(3,11) \oplus \boldsymbol{R}$	$(\wedge^+(U_{(7)})^C)_{l_{3,11}}$	$(W^C_{(7)})_{l_{3,11}}$
(e24)	$\mathfrak{e}^C_8$	$\mathfrak{e}^C_7 \oplus C$	$\mathfrak{P}^C_{\mathfrak{C}}$	C
(e25)	$\mathfrak{f}^C_4$	$\mathfrak{sp}(3, C) \oplus C$	$\mathfrak{P}^C_{\boldsymbol{R}}$	C
(e26)	$\mathfrak{e}^C_6$	$\mathfrak{sl}(6, C) \oplus C$	$\mathfrak{P}^C_{\boldsymbol{C}}$	C
(e27)	$\mathfrak{e}^C_7$	$\mathfrak{so}(12, C) \oplus C$	$\mathfrak{P}^C_{\boldsymbol{H}}$	C
(e28)	$\mathfrak{g}^C_2$	$\mathfrak{sl}(2, C) \oplus C$	$C_3[e_1, e_2]$	C
(e29)	$\mathfrak{e}^C_6$	$\mathfrak{so}(8, C) \oplus C \oplus C$	$\mathfrak{C}^C \oplus \mathfrak{C}^C$	$\mathfrak{C}^C$
(e30)	$\mathfrak{f}^C_4$	$\mathfrak{so}(7, C) \oplus C$	$\mathfrak{C}^C$	$\mathfrak{C}^C_0$
(e31)	$\mathfrak{e}^C_6$	$\mathfrak{sl}(5, C) \oplus \mathfrak{sl}(2, C) \oplus C$	$(\boldsymbol{R}^2 \otimes (\boldsymbol{R}^5)_2)^C$	$(\boldsymbol{R}^5)^C_4$
(e32)	$\mathfrak{e}^C_7$	$\mathfrak{sl}(7, C) \oplus C$	$(\boldsymbol{R}^7)^C_3$	$(\boldsymbol{R}^7)^C_6$
(e33)	$\mathfrak{e}^C_7$	$\mathfrak{so}(10, C) \oplus \mathfrak{sl}(2, C) \oplus C$	$(\boldsymbol{R}^2 \otimes \wedge^+(U_{(5)}))^C$	$W^C_{(5)}$
(e34)	$\mathfrak{e}^C_8$	$\mathfrak{so}(14, C) \oplus C$	$\wedge^+(U_{(7)})^C$	$W^C_{(7)}$

Let
$\mathfrak{J}_F = \{X \in M(3, F) \mid X^* = X\}$
$(F = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{C}', \boldsymbol{H}, \boldsymbol{H}', \mathfrak{C}, \mathfrak{C}')$
be a real Jordan algebra and Der $\mathfrak{J}_F$ the derivation algebra of $\mathfrak{J}_F$. For any $X \in \mathfrak{J}_F$, an endomorphism $\tilde{X}$ of $\mathfrak{J}_F$ is defined by

$$\tilde{X}(Y) = \frac{1}{2}(XY + YX) \quad (Y \in \mathfrak{J}_F).$$

Put
$\mathfrak{J}_{F0} = \{X \in \mathfrak{J}_F \mid \mathrm{tr}X = 0\}$,
$\tilde{\mathfrak{J}}_{F0} = \{\tilde{X} \mid X \in \mathfrak{J}_{F0}\}$,
$\mathfrak{e}_{6F} = \mathrm{Der}\,\mathfrak{J}_F \oplus \tilde{\mathfrak{J}}_{F0}$, $\mathfrak{e}_{7F} = \mathfrak{e}_{6F} \oplus \mathfrak{J}_F \oplus \mathfrak{J}_F \oplus \boldsymbol{R}$,
$\mathfrak{P}_F = \mathfrak{J}_F \oplus \mathfrak{J}_F \oplus \boldsymbol{R} \oplus \boldsymbol{R}$,
$\mathfrak{e}_{8F} = \mathfrak{e}_{6F} \oplus \mathfrak{P}_F \oplus \mathfrak{P}_F \oplus \boldsymbol{R} \oplus \boldsymbol{R} \oplus \boldsymbol{R}$.
Then, it is well known that $\mathfrak{e}_{8F}$ is a real 2-GLA ([4]).

2-GLA's (e1)~(e7) are obtained as $\mathfrak{e}_{8F}$ ($F = \mathfrak{C}', \mathfrak{C}, \boldsymbol{R}, \boldsymbol{C}', \boldsymbol{C}, \boldsymbol{H}', \boldsymbol{H}$). 2-GLA's (e24)~(e27) are obtained by complexification of (e1), (e3), (e4) and (e6), respectively. The 2-GLA (e8) (reap. (e9)) is obtained by the method described in **1.2** from a grade-preserving involution of (e26) (resp. (e27)).

Hereafter, we outline only representations ρ_k of g_0 in Table I.

(2) In case of (e10) and (e28):

Let $\boldsymbol{R}_3[e_1, e_2]$ be a real vector space of all homogeneous polynomials of degree 3 in variables e_1 and e_2. Define a representation ρ of $\mathfrak{sl}(2, \boldsymbol{R})$ on $\boldsymbol{R}_3[e_1, e_2]$ by
$\rho(X)(e_i e_j e_k) = (Xe_i)e_j e_k + e_i(Xe_j)e_k + e_i e_j(Xe_k)$.
In (e10), the representation $\rho_k (k = 1,2)$ of g_0 on g_k is as follows:

$$\rho_1(X, r)\boldsymbol{u} = \rho(X)\boldsymbol{u} + r\boldsymbol{u}, \ \rho_2(X, r)s = 2r.$$

The 2-GLA (e28) is obtained by complexification of (e10).

(3) In case of (e11)~(e16), (e29) and (e30):

In (e29), using automorphisms of $\mathfrak{so}(8, C)$ π and λ which were defined in [2], we define the representation $\rho_k (k = 1,2)$ of g_0 on g_k as follows:

$$\rho_1(X, s, t)(x \oplus y) = (\pi X + s)x \oplus (\lambda\pi X + t)y,$$
$$\rho_2(X, s, t)u = (X + s + t)u.$$

For $F = \mathfrak{C}'$ or $\mathfrak{C}$ put
$F_0 = \{x \in F \mid \mathrm{Re}x = 0\}$,
$F_{00} = \{ia + x \in F^C \mid a \in \boldsymbol{R}, x \in F_0\}$,
where $\mathrm{Re}x$ means the real part of x.

2-GLA's (e11)~(e16) and (e30) are obtained by the method described in **1.2** from (e29).

(4) In case of (e17), (e18), (e31) and (e32):

Let $(\boldsymbol{R}^n)_k := \wedge^k(\boldsymbol{R}^n)$ be the k-th exterior power of $\boldsymbol{R}^n$. Define a representation μ_k of $\mathfrak{sl}(n, \boldsymbol{R})$ on $(\boldsymbol{R}^n)_k$ by
$\mu_k(X)(\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k)$

$$= \sum_{j=1}^{k} \boldsymbol{x}_1 \wedge \cdots \wedge (X\boldsymbol{x}_j) \wedge \cdots \wedge \boldsymbol{x}_k.$$

The representation $\rho_k (k = 1, 2)$ of g_0 on g_k is as follows:

(a) In case of (e17).

$$\rho_1(X, A, r)(a \otimes \boldsymbol{u}) = (Aa) \otimes \boldsymbol{u} + a \otimes (\mu_2(-{}^tX)\boldsymbol{u}) + r(a \otimes \boldsymbol{u}),$$
$$\rho_2(X, A, r)\boldsymbol{x} = (X + 2r)\boldsymbol{x}.$$

(b) In case of (e18).

$$\rho_1(X, r)\boldsymbol{u} = \mu_3(-{}^tX)\boldsymbol{u} + r\boldsymbol{u},$$
$$\rho_2(X, r)\boldsymbol{x} = (X + 2r)\boldsymbol{x}.$$

The 2-GLA (e31) (resp. (e32)) is obtained by complexification of (e17) (resp. (e18)).

(5) In case of (e19)~(e23), (e33) and (e34):

Let $W_{(n)}$ be the $2n$-dimensional real vector space with a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. We define a bilinear from Q on $W_{(n)}$ by

$$Q(e_i, e_j) = 0, \ Q(f_i, f_j) = 0,$$
$$Q(e_i, f_j) = Q(f_i, e_j) = \delta_{ij}.$$

Let $U_{(n)}$ be the subspace of $W_{(n)}$ generated by $\{e_1, \ldots, e_n\}$ and put

$$\wedge^+ U_{(n)} = \sum_{l:\text{even}} \wedge^l U_{(n)}.$$

Let $d\varphi$ be a *half-spin representation* of $\mathfrak{so}(n, n)$ on $\wedge^+ U_{(n)}$. The representation $\rho_k (k = 1, 2)$ of g_0 on g_k is as follows:

(a) In case of (e19).

$$\rho_1(X, A, r)(a \otimes \boldsymbol{u}) = (Aa) \otimes \boldsymbol{u} + a \otimes (d\varphi(X)\boldsymbol{u}) + r(a \otimes \boldsymbol{u}),$$
$$\rho_2(X, A, r)\boldsymbol{x} = (X + 2r)\boldsymbol{x}.$$

(b) In case of (e22).

$$\rho_1(X, r)\boldsymbol{u} = d\varphi(X)\boldsymbol{u} + r\boldsymbol{u},$$
$$\rho_2(X, r)\boldsymbol{x} = (X + 2r)\boldsymbol{x}.$$

The 2-GLA (e33) (resp. (e34)) is obtained by complexification of (e19) (resp. (e22)). 2-GLA's (e20) and (e21) are obtained by the method described in **1.2** from (e33). The 2-GLA (e23) is obtained by the method described in **1.2** from (e34).

Remark. 2-GLA's (e17) and (e31) can not be obtained by Allison's construction.

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