Seminear-rings Characterized by their $\mathcal{S}$-Ideals. I

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1. Introduction and preliminaries. A right seminear-ring is a set R together with two binary operations $+$ and $\cdot$ such that $(R, +)$ and $(R, \cdot)$ are semigroups and for all $a, b, c \in R$: $(a+b)c = ac + bc$ [8, 9]. A natural example of a right seminear-ring is the set $M(S)$ of all mappings on an additively written semigroup S with pointwise addition and composition. A right seminear-ring R is said to have an *absorbing zero* 0 if $a + 0 = 0 + a = a$ and $a \cdot 0 = 0 \cdot a = 0$ hold for all $a \in R$. Throughout this paper, R will denote a right seminear-ring with an absorbing zero. *A seminear-ring homomorphism* between seminear-rings R and R' is a map $\phi : R \to R'$ satisfying $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R$. An *ideal* of a seminear-ring is the kernel of a seminear-ring homomorphism [7]. Generalizing this definition, we call a subset I of a seminear-ring R a *right* (*left*) *$\mathcal{S}$-ideal* if (i) for all $x, y \in I$, $x + y \in I$, and (ii) for all $x \in I$ and $r \in R$, $xr(rx) \in I$. The word *$\mathcal{S}$-ideal* will always mean a subset of R which is both a left and a right $\mathcal{S}$-ideal. If R is a unitary near-ring instead of a seminear-ring then $\mathcal{S}$-ideals of R are just the R-subgroups of R (see [8], p. 14). On the other hand, if R is a unitary ring or more generally, a semiring (in the sense of Golan [6]), then $\mathcal{S}$-ideals are the ideals of R in the usual sense (see [6], p. 55). An element a of a seminear-ring R is (*left*) *distributive* if for all $x, y \in R$, $a(x+y) = ax + ay$; R will be called *distributively generated,* or d.g. for short, if R contains a multiplicative subsemigroup D of (left) distributive elements which generate $(R, +)$. The notion of d.g. seminear-rings provides a common generalization of distributively generated near-rings (cf. [8]) and general semirings (cf. [6]). Lef S be an additively written semigroup and let $E(S)$ be the set of all finite sums $\sum f_i$, where f_i's belong to the set of endomorphisms of S. Then $E(S)$ is a subseminear-ring of $M(S)$ distributively generated by the multiplicative subsemigroup of endomorphisms of S. If A, B, C are three nonempty subsets of a seminear-ring R, then $AB(ABC)$ will denote the set of all finite sums of the form $\sum a_k b_k$ with $a_k \in A$, $b_k \in B$ ($\sum a_k b_k c_k$ with $a_k \in A$, $b_k \in B$, $c_k \in C$). In particular, for each $a \in R$, $aR(Ra)$ will denote the set of all finite sums of the form $\sum ar_k(\sum r_k a)$ with $r_k \in R$. Since R is right distributive, $Ra = \{ra : r \in R\}$. Clearly $aR(Ra)$ is a right (left) $\mathcal{S}$-ideal of R. The $\mathcal{S}$-ideal $aR(Ra)$ will be called the *principal right* (*left*) *$\mathcal{S}$-ideal generated by a*. For any subset S of R, $\langle S \rangle$ will denote the $\mathcal{S}$-ideal generated by S (i.e. the minimal $\mathcal{S}$-ideal containing S). A seminear-ring R is called *right* (*left*) *normal* if $a \in aR(Ra)$ for each $a \in R$; R is *normal* if it is both right and left normal. If A and B are $\mathcal{S}$-ideals of a seminear-ring then the *product* $AB = \{\sum_{k=1}^{n} a_k b_k : a_k \in A,\ b_k \in B\}$ need not admit the structure of an $\mathcal{S}$-ideal. However, if R is a d.g. seminear-ring then an easy calculation shows that AB is an $\mathcal{S}$-ideal of R. We now define the sum of $\mathcal{S}$-ideals of a seminear-ring R in the following way. For $\mathcal{S}$-ideals A, B of R, the *sum* $A + B$ is defined by the set of all finite sums $\sum x_i$ where each $x_i = (a_i + b_i)$ with $a_i \in A$ and $b_i \in B$. More generally, if $\{A_i : i \in I\}$ is an arbitrary family of $\mathcal{S}$-ideals of a seminear-ring R, then the *sum* $\sum_{i \in I} A_i$ is the set of all finite sums $\sum x_j$ where $x_j = \sum_{i \in I} a_{ij}$ such that $a_{ij} \in A_i$ and $a_{ij} = 0$ for all except finitely many $i \in I$. If R is a d.g. seminear-ring then it is easily verified that for $\mathcal{S}$-ideals A, B of R, $A + B$ is the unique minimal member of the family of all $\mathcal{S}$-ideals of R containing both A and B. More generally, if $\{A_i : i \in I\}$ is a family of $\mathcal{S}$-ideals, then $\sum_{i \in I} A_i$ is the unique minimal member of the family of all $\mathcal{S}$-ideals of R containing the $\mathcal{S}$-ideals $\{A_i : i \in I\}$. Moreover $\bigcap_{i \in I} A_i$ is the unique maximal member of the family of all $\mathcal{S}$-ideals of R contained in the $\mathcal{S}$-ideals $\{A_i : i \in I\}$.

A ring R is *fully idempotent* if each ideal I of

R is idempotent, that is, if $I = I^2$. Several characterizations of these rings were given by Courter [4] and they play an important role in the study of (von Neumann) regular and V-rings [5], both of which are proper subclasses of fully idempotent rings. In this paper we initiate the study of a class of seminear-rings which are analogous to fully idempotent rings. We call them *strongly idempotent*, or for short SI-seminear-rings. In section 2 we describe a characterization of these seminear-rings. We also discuss some properties of regular seminear-rings (definition follows) which are an important subclass of SI-seminear-rings. Below we announce our main results. The details will appear elsewhere.

2. *SI* and regular seminear-rings. Definition 1. *A seminear-ring R will be called strongly idempotent or for short, SI-seminear-ring if for each $\mathcal{S}$-ideal I of R, $I = \langle I^2 \rangle$.*

Thus if R is a unitary ring which is SI in the above sense, then R is precisely a fully idempotent ring. The following proposition characterizes SI-seminear-rings.

Proposition 1. *The following assertions are equivalent*:

(1) *R is SI*

(2) *For each pair of $\mathcal{S}$-ideals A, B of R, $A \cap B = \langle AB \rangle$.*

(3) *$\mathcal{S}$-ideals of R form a semilattice $(\mathcal{L}_R, \Lambda)$ with $A \Lambda B = \langle AB \rangle$ for each pair of $\mathcal{S}$-ideals A, B of R.*

Extending the definition of a regular near-ring ([8], p. 345), a seminear-ring R will be called *regular* if for each $a \in R$, $\exists\, b \in R$ such that $aba = a$. The following corollary is an immediate consequence of the above proposition.

Corollary 1. *If R is a regular seminear-ring then R is SI with $I = I^2$ for each $\mathcal{S}$-ideal I of R.*

The following characterization of regular seminear-rings can be proved by using the usual techniques ([8], p. 345).

Proposition 2. *A seminear-ring R is regular if and only if R is normal and each principal left $\mathcal{S}$-ideal is generated by an idempotent.*

Corollary 2. *If R is a regular seminear-ring with central idempotents, then R is reduced* (*that is, having no nonzero nilpotent elements*).

Corollary 3. *A regular seminear-ring contains no nonzero nil $\mathcal{S}$-ideals* (*An $\mathcal{S}$-ideal is nil if each of its elements is nilpotent*).

Proposition 3. *Let R be a seminear-ring with central idempotents. Then the following assertions are equivalent*:

(1) *R is regular.*

(2) *R is normal, each principal right $\mathcal{S}$-ideal $aR = \{ar : r \in R\}$, and for each right $\mathcal{S}$-ideal K and left $\mathcal{S}$-ideal L, $K \cap L = KL$.*

Next we construct a class of examples of SI-seminear-rings which are neither regular nor $\mathcal{S}$-simple (that is, having no nonzero proper $\mathcal{S}$-ideals). First we outline a procedure for constructing seminear-rings:

Let R be a right near-ring and $(S, \cdot)$ be a semigroup, and let $\phi : (S, \cdot) \rightarrow (R, \cdot)$ be a homomorphism. Let $A = S \cup R$. On the set A define the binary operations $+$ and $\cdot$ as follows:

$$r + s = r + \phi(s)$$
$$s + r = \phi(s) + r$$
$$rs = r\phi(s)$$
$$sr = \phi(s)r \quad \text{for all } r \in R,\ s \in S.$$

Moreover, $r_1 + r_2 =$ (sum in R) ; $r_1 \cdot r_2 =$ (product in R) for all $r_1, r_2 \in R$; $s_1 + s_2 = \phi(s_1) + \phi(s_2)$ and $s_1 \cdot s_2 =$ (product in S) for all $s_1, s_2 \in S$. It is easily verified that $(A, +, \cdot)$ is a seminear-ring.

Example 1. Let S be a semigroup with identity e, and let C denote the bicyclic semigroup, that is, $C = N_0 \times N_0$, where N_0 is the set of non-negative integers and the multiplication in C is defined by $(m, n)(p, q) = (m + p - \min(n, p), n + p - \min(n, p))$. It is well known (see for example, [3], 1.12) that C is a bisimple inverse monoid with $(m, n)^{-1} = (n, m)$ and identity (0,0). Let $T = C \times S$ with the following multiplication:

$((m, n), s)((p, q), t) = ((m, n)(p, q), f(n, p))$ where $f(n, p) = s$, t or st according to whether $n > p$, $n < p$ or $n = p$. It can be verified that T is a simple semigroup (that is, a semigroup with no proper (semigroup) ideals). Furthermore it can be shown that T is a regular semigroup (that is, $x \in xTx$ for all $x \in T$) if and only if S is regular. Now let $T = C \times S$ where S is a non-regular monoid (for example, $S = (N, \cdot)$) and let $R = \{0,1\}$ be the near-ring defined by the following tables:

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	1	1

Let $\phi : (T, \cdot) \to (R, \cdot)$ be the homomorphism defined by $\phi(t) = 0$ for all $t \in T$. Let $A = T \cup R$. On the set A, define $+$ and $\cdot$ as described by the above procedure. Then $(A, +, \cdot)$ is a seminear-ring. Now let I be an $\mathscr{S}$-ideal of A. Then $\{0,1\} \subseteq I$. Suppose $I \neq \{0,1\}$. Then $I \setminus \{0,1\}$ is a semigroup ideal of T. Hence for $b \in T$ and $c \in I \setminus \{0,1\} \subseteq T$, bc and $cb \in I \setminus \{0,1\}$. Since T is a simple semigroup, so T has no proper ideals. Therefore $I \setminus \{0,1\} = T$. Hence $I = T \cup \{0,1\}$. Thus $I = A$. Now adjoin an element θ to A where $\theta \notin A$ such that $a + \theta = \theta + a = a$, for all $a \in A \cup \theta$, and $a\theta = \theta a = \theta$, for all $a \in A \cup \theta$. Now put $A \cup \theta = A'$. Then $(A', +, \cdot)$ is a seminear-ring with an absorbing zero θ. Now let I be any $\mathscr{S}$-ideal of $(A', +, \cdot)$. If $I = \{\theta\}$, then clearly $I = I^2$. Suppose $I \neq \theta$. Then $I \setminus \{\theta\}$ is an $\mathscr{S}$-ideal of the seminear-ring A. Hence as shown above, $I \setminus \{\theta\} = A$ or $I \setminus \{\theta\} = \{0,1\}$. Thus $I = A'$ or $I = \{\theta\} \cup \{0,1\} = \{0, \theta, 1\}$. Thus in any case, we have $I = I^2$. Hence A' is an SI-seminear-ring. Clearly A' is not regular. Moreover, A' is not $\mathscr{S}$-simple since $\{0, \theta, 1\}$ is a proper $\mathscr{S}$-ideal.

Finally we state the following embedding theorem for seminear-rings which may be of independent interest.

Theorem 1. *Let R be an arbitrary ring. Then R can be embedded in a regular seminear-ring with an absorbing zero and a multiplicative identity.*

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