The Degree Function for Cellular Dynamics

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(Communicated by Kiyosi ITÔ, M. J. A., Jan. 12, 1995)

Abstract: To study the orbit structures of cellular dynamics, one have to study the entropy decreasing factor maps over some sofic systems. It is difficult to analyze the structure of these factor maps because the inverse image of almost every point is uncountable. In this paper, the author proposes a function called *the degree function,* a generalization of the degree of factor maps in finite to one cases, which indicates the exponential rate of the number of the inverse image of a word. Using the degree function, we get upper bounds of decreasing in spatial entropies and some relations.

1. Introduction and the background. It is well known that the cellular automata contains various orbit behaviors although its definition is simple. Given

Lattice. L, often using D-dimensional lattice $\boldsymbol{Z}^D$.

Cellular state space. fix a finite set (alphabet) A, for example $\{0,1\}$

Configuration space. $X = A^L$, or a sofic system (see section 2)

Neighborhood. a finite set Λ of lattice containing the origin

Interaction. a map $f : A^\Lambda \mapsto A$ (called a local map)

its dynamics $\tau : X \to X$ is defined $(\tau x)_s = f(x_t : t \in \sigma_s \Lambda)$ for all $x \in X$ where σ_t is the translation from origin to $t \in L$.

The notion of cellular automata has been recognized as an important model of "self organization". In the late forties J. von Neumann introduced the "29 states self reproducing automata", which is the origin of this stream [4].

But the universe seems to be more attractive from the viewpoint of dynamical systems theory. In the early eighties S. Wolfram [5] developed his numerical research on one-dimensional cellular automata as the target of dynamical systems and statistical mechanics. However, not so many results are obtained from mathematical viewpoint, especially from ergodic theory. In the present report, we call cellular automata over one-dimensional lattice $\boldsymbol{Z}$ as *cellular dynamics.*

The cellular dynamics are continuous maps with shift commuting property. Those are *factor maps* over sofic systems. When a factor map is surjective on the configuration space, it is well known that the factor map is boundedly finite to one [2]. These cases are deeply studied from the viewpoint of the isomorphism problem between topological Markov shifts [3].

But the case of not surjective, the topological entropy decreases. In these cases, the factor maps are uncountably infinite to one [2]. It is very difficult to use the standard tools that we have already known on maps of intervals and so on. In the present paper we introduce the degree function which is the number of n-word's inverse images. Then there are some relations between the degree function and spatial entropies as shown in section 4. We announce the results and the proofs will be published elsewhere.

The author would like to thank Professor Yoichiro Takahashi for fruitful discussions and constant encouragement, and Professors Michiko Yuri and Satoshi Takahashi for their helpful advice.

2. Notations. Let A be a finite set and σ the *shift transformation* on $A^{\boldsymbol{N}} = \{x = (x_n)_{n \in \boldsymbol{N}} ; x_n \in A\}$, i.e., $(\sigma x)_n = x_{n+1}$ $(n \in \boldsymbol{N})$. The shift transformation on $A^{\boldsymbol{Z}}$ defined in a similar way. A pair (X, σ) consisting of a σ-invariant set X and the restriction of σ to X, denoted again by σ, is called a (one-sided or two-sided) shift. If A is endowed with a topology and X is compact, then it is called a topological shift or a subshift. The following notations are used in this paper:

$W_n(X) = \{x_0x_1x_2 \cdots x_{n-1}; x = (x_n) \in X\}$ (n-word set),

$W(X) = \bigcup_{n=1}^{\infty} W_n(X)$ (word set of X).

For a word $w = a_1 \cdots a_n \in A^n$, $|w| = n$ (length of w), $[w] = \{x \in X ; x_0 \cdots x_{n-1} = w\}$ (cylinder set). For a subset W of the union $\cup_{n=1}^{\infty} A^n$,

$[W] = \bigcup_{w \in W} [w]$,

$M(W) = \{x = (x_n) ; (x_{i+n})_{n \geq 0} \in [W]$ for any $i\}$.

If a shift (X, σ) satisfies $X = M(W)$ with $W = W_{p+1}(X)$ then, (X, σ) is called a *p-Markov shift* or simply a *Markov shift.*

Let (X, σ) be a subshift and n be a positive integer. Then a subshift $(X^{[n]}, \sigma)$ is called its *higher block system* of *n-block system* if $X^{[n]}$ is defied by

$X^{[n]} = \{(x_i, \ldots, x_{i+n-1})_{i \in \mathbf{Z}} ; (x_i)_{i \in \mathbf{Z}} \in X\}$,

and $(X^{[n]}, \sigma)$ is topologically conjugate to (X, σ) for all $n \in \mathbf{N}$.

Now, the shift homomorphisms are introduced, which are the shift-commuting continuous maps and are often called *factor maps.*

Theorem 2.1. [2] *Let (X, σ) and (Y, σ) be subshifts over alphabets A_X and A_Y respectively, and a map $\tau : X \to Y$ be continuous and assume $\tau\sigma = \sigma\tau$. Then there exist a finite interval $\Lambda = \{k, \ldots, k + p\}$ and a map $f : A_X^{p+1} \mapsto A_Y$ such that $(\tau x)_i = f(x_{i+k}, \cdots, x_{i+k+p})$.*

For each $n \in \mathbf{N}$, the n-block map $f_n : A_X^{p+n} \mapsto A_Y^n$ is defined from the local map f so that $f_n(w_1 \cdots w_{p+n}) = f(w_1 \cdots w_{p+1}) f(w_2 \cdots w_{p+2}) \cdots f(w_n \cdots w_{p+n})$. Taking higher-block systems, if necessary, any cellular dynamics can always be assumed to be *one-block* defined through a local map with $p = 0$ in Theorem 2.1.

A subshift (X, σ) over an alphabet A is called a *sofic system* if there is a Markov shift (Σ, σ) (called a *Markov cover* of (X, σ)) and a surjective factor map $\pi : \Sigma \to X$. The factor images of sofic systems are also sofic systems.

3. Entropy and Gibbs measure. Let (X, σ) be a subshift, μ be a σ-invariant probability measure and α be a finite measurable partition of X. The metrical entropy for (X, σ, μ, α) is given by $h_\mu(X, \sigma, \alpha) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_n)$ where $H_\mu(\alpha) = -\sum_{A \in \alpha} \mu(A) \log \mu(A)$ and α_n denotes the refinement $\alpha \vee \sigma^{-1}\alpha \vee \cdots \vee \sigma^{-(n-1)}\alpha$. The metrical entropy for (X, σ, μ) is defined by $h_\mu(X, \sigma) = \sup_\alpha h_\mu(X, \sigma, \alpha)$. For two invariant probability measures μ, ν on (X, σ), the relative entropy are defined if the limit exists as follows:

$$h(\mu \mid \nu) = \lim_{n\to\infty} \frac{1}{n} \sum_{C \in \alpha_n} \nu(C) \log \frac{\mu(C)}{\nu(C)}.$$

Now, we introduce a class of measures called Bowen type Gibbs measures following [1]. At first, we think of topologically mixing Markov shifts, i.e., aperiodic Markov shifts (X, σ).

Let $C(X)$ be the set of all continuous functions on X and for $U \in C(X)$, we denote $\mathrm{var}_k(U) = \sup_{x,y \in X}\{|U(x) - U(y)| ; x_i = y_i, 0 \leq i < k\}$ and $F(X) = \{U \in C(X) ; \mathrm{var}_k(U) \leq ba^k, b \geq 0, 0 \leq a < 1, k \geq 0\}$.

Theorem 3.1. [1] *For a function $U \in F(X)$, there exists an unique shift invariant measure μ_U such that*

$$C_1 \leq \frac{\mu_U([x_0 \cdots x_{m-1}])}{\exp(-mP(U) - S_m U(x))} \leq C_2 \text{ and}$$

$$P(U) = h_{\mu_U}(X, \sigma) - \int_X U d\mu_U$$

for some constants $C_1 > 0$, $C_2 > 0$ and $S_n U(x) = \sum_{k=0}^{n-1} U(\sigma^k x)$.

We get the same result for topologically mixing sofic systems owing to the existence of the topologically mixing Markov cover.

4. Main results. In this section, we assume that factor maps are one-block and that sofic systems are one-sided without loss of generality.

Definition 4.1. Let (X, σ) and (Y, σ) be two sofic systems and τ be a surjective cellular dynamics from X to Y. Let $d(w) = \# f_{|w|}^{-1}(W)$ and we call it *the degree function.*

Proposition 4.2. *There exists the limit*

$$h_d(y) = \lim_{n\to\infty} \frac{1}{n} \log d(y_0 y_1 \cdots y_{n-1}).$$

for almost every $y \in Y$ with respect to any shift invariant probability Borel measures. Moreover, the above limit is independent of the choice of the local map f.

Theorem 4.3. *Let μ be a measure on X which attain the maximal entropy, and $\nu = \tau^*\mu$ be the induced measure by τ. Suppose $h_d(y) = 0$ for ν almost every y in Y. Then τ is boundedly finite to one, i.e., $\sup_{y \in Y} \# \tau^{-1}(y) < \infty$.*

Theorem 4. 4. *Let (X, σ) and (Y, σ) be two sofic systems and $\tau : X \mapsto Y$ be a surjective cellular* *dynamics. Take an invariant measure* μ *and set* $\nu = \tau^*\mu$. *Then the following inequality holds*:

$$h_\mu(X, \sigma) - h_\nu(Y, \sigma) \le \int_Y h_d d\nu.$$

Theorem 4.5. *Let* (X, σ) *and* (Y, σ) *be topologically mixing sofic systems and* $\tau : X \mapsto Y$ *be a surjective cellular dynamics. Take* $U \in F(Y)$ *and let* $V = \tau_* U \in F(X)$. *If* $\mu = \mu_V$ *and* $\nu = \tau^*\mu$ *then the following equality holds*:

$$h_\mu(X, \sigma) = h_\nu(Y, \sigma) + \int_Y h_d d\nu.$$

Theorem 4.6 (Gibbs type variational principle). *Under the same assumption as in Theorem* 4.5, *the following equalities hold*:

$$P(V) - P(U) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w \in W_n(Y)} d(w)\mu_U([w])$$

$$= h(\nu \mid \mu_U) + \int_Y h_d d\nu$$

$$= \max_{\mu:\sigma-inv.} \{h(\mu \mid \mu_U) + \int_Y h_d d\mu\}.$$

Theorem 4.7. *Let* w *be a word and* y *be the periodic point defined by* $y = www\cdots \in A^{\boldsymbol{N}}$. *Then* $h_d(y) = |w|^{-1} \log \lambda(w)$ *where* $\lambda(w)$ *is the maximal eigenvalue for a non-negative integer matrix* $A(w)$ *associated with the local map.*

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