23. *Algebraic Geometry of Center Curves in the Moduli Space of the Cubic Maps*

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0. Introduction. In our previous paper [6], we have defined the so-called center curves BC_p and CD_p, which are algebraic curves, for the real cubic maps. The attached figure 1 gives the graphs of these curves for $p = 1, 2, 3, 4$. Note that these graphs exist only in the first and third quadrants. The same holds also for other values $p = 5, 6, \cdots$.

In the present paper we consider the complex maps. For such a cubic map g, we have two normal forms; $x^3 - 3Ax \pm \sqrt{B}$, $A, B \in \mathbf{C}$. Therefore, the complex affine conjugacy class of g can be represented by (A, B). The moduli space, consisting of all affine conjugacy classes of cubic maps, can be identified with the coordinate space $\mathbf{C}^2 = \{(A, B)\}$. For the post-critically finite complex cubic maps, the **center curves** CD_p, BC_p can be defined in the same way as in [6]. In section 1, we show how the equations of these curves are obtained by induction on p.

We can embed $\mathbf{C}^2$ canonically in $\mathbf{P}^2(\mathbf{C}) : (A, B) \to (1 : A : B)$. Then an affine algebraic curve $V_0 = \{(A, B) \in \mathbf{C}^2 : h(A, B) = 0\}$ uniquely determines a projective algebraic curve $V = \{(C : A : B) \in \mathbf{P}^2(\mathbf{C}) : H(C : A : B) = 0\}$ in $\mathbf{P}^2(\mathbf{C})$ such that $h(A, B) = H(1 : A : B)$ and $V \cap \mathbf{C}^2 = V_0$.

Definition. For a center curve V_0, the corresponding projective algebraic curve V is called the **projective center curve**. We denote by PBC_p and PCD_p, these curves corresponding to BC_p and CD_p respectively.

In sections 2 and 3, we give some properties of these curves from the viewpoint of algebraic geometry ([1]).

1. The equations of center curves. Let $f(x) = x^3 - 3Ax + \sqrt{B}$, with critical points $\pm\sqrt{A}$.

The equation of curve BC1 is obtained as follows:

$$f(\sqrt{A}) - (-\sqrt{A}) = (-2A + 1)\sqrt{A} + \sqrt{B} = 0$$
$$f(-\sqrt{A}) - \sqrt{A} = (2A - 1)\sqrt{A} + \sqrt{B} = 0.$$

Therefore,

$$\mathrm{BC1} : B = A(2A - 1)^2.$$

The equation of curve CD1 is obtained as follows:

$$f(\sqrt{A}) - \sqrt{A} = (-2A - 1)\sqrt{A} + \sqrt{B} = 0,$$
$$f(-\sqrt{A}) - (-\sqrt{A}) = (2A + 1)\sqrt{A} + \sqrt{B} = 0.$$

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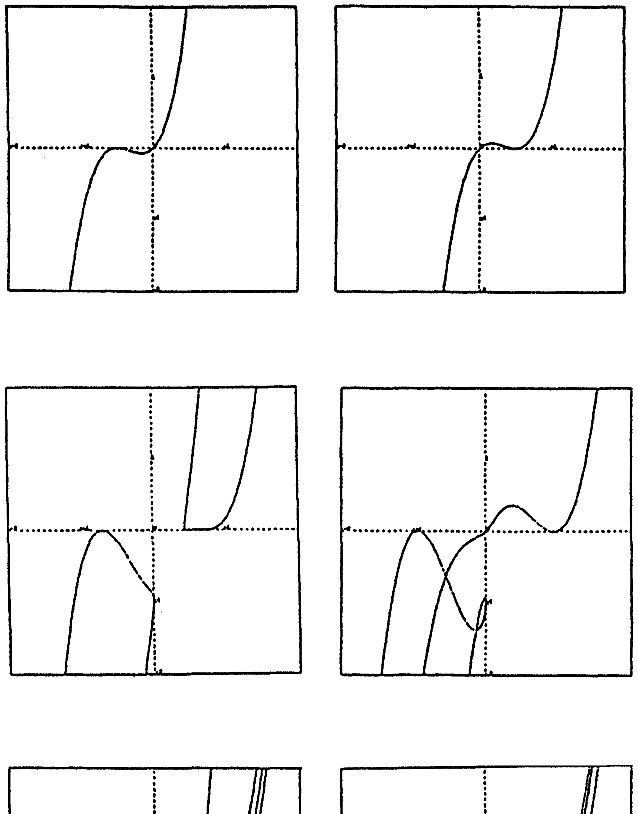

Fig. 1

Therefore,

$$\mathrm{CD1} : B = A(2A+1)^2.$$

The equation of curve BC2 is obtained as follows:

$$f^2(\sqrt{A}) - (-\sqrt{A}) = (-8A^4 + 6A^2 + 1 - 6AB)\sqrt{A} + (12A^3 - 3A + 1 + B)\sqrt{B} = 0,$$
$$f^2(-\sqrt{A}) - \sqrt{A} = (8A^4 - 6A^2 - 1 + 6AB)\sqrt{A} + (12A^3 - 3A + 1 + B)\sqrt{B} = 0.$$

Therefore,

$$\mathrm{BC2} : B^3 - 12A^3B^2 - 6AB^2 + 2B^2 + 48A^6B + 24A^3B + 21A^2B - 6AB + B - 64A^9 + 96A^7 - 20A^5 - 12A^3 - A = 0.$$

The equation of curve CD2 is obtained as follows:

$$f^2(\sqrt{A}) - \sqrt{A} = (-8A^4 + 6A^2 - 1 - 6AB)\sqrt{A} + (12A^3 - 3A + 1 + B)\sqrt{B} = 0,$$
$$f^2(-\sqrt{A}) - (-\sqrt{A}) = (8A^4 - 6A^2 + 1 + 6AB)\sqrt{A} + (12A^3 - 3A + 1 + B)\sqrt{B} = 0.$$

Thus

$$B(12A^3 - 3A + 1 + B)^2 - A(-8A^4 + 6A^2 - 1 - 6AB)^2 = 0.$$

Fixed points can be also considered as periodic points of period 2. So, this curve contains CD1. Dividing the left-hand side of the last equation by the defining polynomial of CD1, we get the equation of CD2 as follows:

$$\mathrm{CD2} : B^2 - 8A^3B + 4A^2B - 5AB + 2B + 16A^6 - 16A^5 - 12A^4 + 16A^3 - 4A + 1 = 0.$$

Suppose now,

$$f^p(\sqrt{A}) = P_p\sqrt{A} + Q_p\sqrt{B},$$
$$f^p(-\sqrt{A}) = -P_p\sqrt{A} + Q_p\sqrt{B},$$

where P_p, Q_p are polynomials of A, B. Then we have

$$P_p = AP_{p-1}^3 + 3BP_{p-1}Q_{p-1}^2 - 3AP_{p-1},$$
$$Q_p = 3AP_{p-1}^2Q_{p-1} + BP_{p-1}^3 - 3AQ_{p-1} + 1.$$

The equation of curve BC_p is obtained as follows:

$$f^p(\sqrt{A}) - (-\sqrt{A}) = (P_p + 1)\sqrt{A} + Q_p\sqrt{B} = 0,$$
$$f^p(-\sqrt{A}) - \sqrt{A} = (-P_p - 1)\sqrt{A} + Q_p\sqrt{B} = 0.$$

Therefore,

$$\mathrm{BC}_p : (P_p + 1)^2A - Q_p^2B = 0.$$

The equation of curve CD_p is obtained as follows:

$$f^p(\sqrt{A}) - \sqrt{A} = (P_p - 1)\sqrt{A} + Q_p\sqrt{B} = 0,$$
$$f^p(-\sqrt{A}) - (-\sqrt{A}) = (-P_p + 1)\sqrt{A} + Q_p\sqrt{B} = 0.$$

Let

$$\tilde{\phi}_p(A, B) := (P_p - 1)^2A - Q_p^2B.$$

If $\phi_q(A, B) = 0$ is the defining equation of CD_q, then we have

$$\tilde{\phi}_p(A, B) = \prod_{q|p} \phi_q(A, B).$$

Therefore if $\{q_1, \cdots, q_n\}$ is the set of all divisors of p except p, then

$$\mathrm{CD}_p : \phi_p(A, B) = \tilde{\phi}_p(A, B) / \prod_{i=1}^{n} \phi_{q_i}(A, B) = 0.$$

2. The intersection with the line at infinity. Suppose p is given.

q_i $(i = 1, \cdots, n)$ will have the same meaning as above. From the preceeding paragraph, we obtain easily the following lemma.

Lemma. (a) *Suppose the defining equation* $\phi(A, B)$ *of* CD_p *is*

(1) $\phi(A, B) = \phi_k(A, B) + \phi_{k-1}(A, B) + \cdots + \phi_0(A, B) = 0,$

where $\phi_i(A, B)$ *is a homogeneous polynomial of degree* i $(i = 0, \cdots, k)$. *Then* $\phi_k(A, B) = \alpha A^k$ (α *is constant*) *and* $k = 3^p - \sum_{i=1}^n \mu(q_i)$, *with* $\mu(q_i)$ *is the total degree of* CD_{q_i}.

(b) *Let now,*

(2) $\phi(A, B) = \psi_m(A)B^m + \psi_{m-1}(A)B^{m-1} + \cdots + \psi_0(A) = 0.$

Then $\psi_m(A)$ *is constant and* $m = 3^{p-1} - \sum_{i=1}^n \nu(q_i)$ *with* $\nu(q_i)$ *is the degree of* CD_{q_i} *with respect to* B. *Moreover, the inequalities* $\mu(q_i) > \nu(q_i)$ *and* $k > m$ *are always satisfied.*

(c) *If we decompose the defining polynomial of* BC_p *like* (1), (2), *we obtain the highest term* βA^k (β *is constant*), $k = 3^p$ *as the term corresponding to* $\phi_k(A, B)$ *in* (1), *and constant* $\times B^m$, $m = 3^{p-1}$ *as the term corresponding to* $\psi_m(A, B) \times B^m$ *in* (2).

We obtain the following theorem from the above lemma.

Theorem 1. *Each projective center curve and the line at infinity,* $L_\infty : C = 0$, *intersect at the point* $(0:0:1)$ *only. This point* $(0:0:1)$ *is singular and its multiplicity can be calculated explicitly by the integer* p.

Proof. It is sufficient to consider the (C, A) affine part of each projective center curve. Each (C, A) affine part of PCD_p and PBC_p are, respectively, $C^d + \sum_{i=d+1}^N \phi_i(A, C)$ and $C^e + \sum_{i=e+1}^N \psi_i(A, C)$, where ϕ_i and ψ_i are homogeneous polynomials of degree i, $d = 2 \cdot 3^{p-1} - \sum_{i=1}^n (\mu(q_i) - \nu(q_i))$, and $e = 2 \cdot 3^{p-1}$. Therefore, for PCD_p (resp. PBC_p), $(0:0:1)$ is singular with multiplicity d (resp. e).

Remark. PCD1 and PBC1 are both cuspidal cubic. Bul for $p \geq 2$, the point $(0:0:1)$ is not a "simple cusp", because of the difference between the degree of the highest term containing A and that rf C. For the definition of "simple cusp", see [2]. Morcover, it has only one tangent line L_∞.

3. Case $p = 1,2$. We get the following theorem about the irreducibility of each projective center curve, which is based on Kaltofen's algorithms on *risa-asir* (computer algebra system) ([4]).

Theorem 2. *Projective center curves* $PCDi$ *and* $PBCi$ $(i = 1,2)$ *are irreducible.*

We obtain the estimate for genus g of each projective center curve Γ, using the following well-known lemma:

Lemma ([3]). *Let* Γ *be an irreducible curve of degree* n. *Let* Sing $\Gamma = \{P_1, \cdots, P_k\}$ *be the set of singular points* P_i *of* Γ. *Let* r_i *be the multiplicity of* P_i. *Then,*

$$g \leq \frac{(n-1)(n-2)}{2} - \sum_{i=1}^k \frac{r_i(r_i - 1)}{2}.$$

Theorem 3. *The curves* $PCD1$ *and* $PBC1$ *are rational. The genus of* $PCD2$ *is not greater than* 3. *The genus of* $PBC2$ *is not greater than* 9.

Proof. We can express

$$\mathrm{PCD}_p = \mathrm{CD}_p \cup (L_\infty \cap \mathrm{PCD}_p) = \mathrm{CD}_p \cup \{(0:0:1)\}.$$

The same decomposition holds for PBC_p.

PCD2 is of degree 6. It has one 4-fold point $(0:0:1)$ and one ordinary double point $(0.25, -0.4375)$. Therefore, $g \leq 3$. PBC2 is of degree 9. It has one 6-fold point $(0:0:1)$ and four ordinary double points as follows:

$$(-0.1341351918179714, -1.37344484910264),$$
$$(-0.5531033117555605, -0.6288238268413773),$$
$$(0.3436192517867655 + 0.3041906503790061 * i, \quad 0.6886343379400248 - 0.04267412324347224 * i),$$
$$(0.3436192517867655 - 0.3041906503790061 * i, \quad 0.6886343379735695 + 0.04267412329900053 * i).$$

Therefore, $g \leq 9$.

We would like to state the follwing conjecture.

Conjecture for projective center curves. *All projective center curves are irreducible. All singular points except* $(0:0:1)$ *are ordinary double points.*

References

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