19. *Complete Local* (S_{n-1}) *Rings of Type* $n \geq 3$ *are Cohen-Macaulay**)

By Yoichi AOYAMA

Department of Mathematics Education, Shimane University

(Communicated by Heisuke HIRONAKA, M. J. A., March 14, 1994)

§1. Introduction. Let A be a local ring of dimension d with maximal ideal m. The *type* of A, denoted by $r(A)$, is defined to be the dimension of $\mathrm{Ext}_A^d(A/m, A)$ as a vector space over A/m. Then Gorenstein local rings are characterized as Cohen-Macaulay local rings of type one (Bass [1]). Vasconcelos [12, p.53] conjectured that the condition $r(A) = 1$ is sufficient to imply that A is Gorenstein (cf. [4, p. 30]). Foxby [4] proved this conjecture for local rings containing a field, for unmixed local rings and for local rings satisfying some other conditions (along with a conjecture for modules). The conjecture was proven in general by Roberts [9], using a minimal free resolution of a dualizing complex. By modifying Roberts' argument, Costa, Huneke and Miller [3] proved that complete local domains of type two are Cohen-Macaulay. They also showed that there exists a non-Cohen-Macaulay equidimensional complete local ring of type two and that there is a non-Cohen-Macaulay reduced complete local ring of type two. Improving their method, Marley [6] proved that unmixed local rings of type two are Cohen-Macaulay and asked if complete local (S_{n-1}) rings of type $n \geq 3$ are Cohen-Macaulay. Kawasaki [5] answered Marley's question in the affirmative for local rings containing a field, making use of Theorem 3 in Bruns [2]. In this note we show that the question has the affirmative answer in general, using Kawasaki's idea. We also give a generalization for modules corresponding to that in [5].

§2. Results. Let R be a commutative noetherian ring. For an R-module M and a prime ideal p, the i-th *Bass number* of M at p, denoted by $\mu^i(p, M)$, is defined to be $\lambda(\mathrm{Ext}_R^i(R/p, M)_p)$, where λ denotes length. Let $I^{\cdot}$ be a minimal injective resolution of M. Then $\mu^i(p, M)$ is equal to the number of copies of $E(R/p)$ which appear in I^i as a direct summand, where $E(R/p)$ denotes the injective envelope of R/p. For basic properties of Bass numbers, see Bass [1]. Let t be an integer. A finitely generated R-module M is said to be (S_t) if depth $M_p \geq \min\{t, \dim M_p\}$ for every p in $\mathrm{Supp}(M)$. In the following A always denotes a local ring of dimension d with maximal ideal m. For an A-module M, $\mu^i(m, M)$ is called the i-th *Bass number* of M and denoted by $\mu^i(M)$. Let M be a finitely generated A-module of dimension s. The *type* of M, denoted by $r(M)$, is defined to be $\mu^s(M)$. Let p be in $\mathrm{Supp}(M)$. If $\dim M_p + \dim A/p = \dim M$, then $r(M_p) \leq r(M)$ by [4, Theorem (5.1)] or [8, Proposition II. 4.1]. We here note that if A is (S_2) and

*) Dedicated to Professor Tomoharu Akiba on his sixtieth birthday.

catenary, then $\dim A/p = d$ for every associated prime ideal p of A (cf. [7, p. 38]). For definitions and basic facts on homological invariants, minimal free resolutions and dualizing complexes, we refer the reader to Roberts [8]. We first recall a version of [2, Theorem 3] for local rings which do not necessarily contain a field.

Theorem 1 ([2, Theorem 3 and Remark (b)]). *Consider a complex*

$$0 \to F_s \xrightarrow{f_s} F_{s-1} \to \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0$$

of free A-modules of finite rank with $F_s \neq 0$ and $f_i(F_i) \subseteq mF_{i-1}$ for $i = 1, \ldots, s$. Put $r_j = \sum_{i=j}^{s}(-1)^{i-j} \operatorname{rank} F_i$ and let I_j be the ideal generated by the r_j-minors of (a matrix representing of) f_j for $j = 1, \ldots, s$. Suppose that for some positive integer t, $\dim A/I_j \leq d - t - j$ for $j = 1, \ldots, s$. Then $r_j \geq t - 1 + j$ for $j = 1, \ldots, s-1$.

For a finitely generated A-module M, $\hat{M}$ denotes the m-adic completion of M. The main result is as follows.

Theorem 2. *Let $n \geq 3$ be an integer. If $r(A) \leq n$ and $\hat{A}$ is (S_{n-1}), then A is Cohen-Macaulay.*

Proof. We proceed by induction on $d = \dim A$. We may assume that A is complete, and hence that A has a dualizing complex. Suppose that A is not Cohen-Macaulay and let $t = \operatorname{depth} A$. Then $d > t \geq n - 1$. By the induction hypothesis, A_p is Cohen-Macaulay for every prime ideal $p \neq m$. Let $D.$ be a dualizing complex of A, where $D_i = \bigoplus \{E(A/p) \mid p \in \operatorname{Spec}(A), \dim A/p = i\}$, and let $F.$ be a minimal free resolution of D.

$$\begin{array}{ccccccccccccc}
F.: \cdots & \longrightarrow & F_{d+1} & \xrightarrow{f_d} & F_d & \xrightarrow{f_{d-1}} & \cdots & \xrightarrow{f_t} & F_t & \longrightarrow & 0 \cdots & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & & & \\
D.: \cdots & \longrightarrow & 0 & \longrightarrow & D_d & \longrightarrow & \cdots & \longrightarrow & D_t & \to \cdots \longrightarrow & D_0 & \longrightarrow & 0 \cdots
\end{array}$$

We have rank $F_i = \mu^i(A)$ for every i by [8, Theorem II. 3.6]. For every prime ideal $p \neq m$, $H_i(F.)_p = 0$ for $i \neq d$ because A_p is Cohen-Macaulay. Hence the complex $(F_d \to \cdots \to F_t \to 0) \otimes A_p$ is exact and split for every prime ideal $p \neq m$. Set $G_i = \operatorname{Hom}_A(F_{d-i}, A)$ and $g_i = {}^t f_{d-i}$, and consider the complex

$$G.: 0 \longrightarrow G_{d-t} \xrightarrow{g_{d-t}} G_{d-t-1} \longrightarrow \cdots \xrightarrow{g_2} G_1 \xrightarrow{g_1} G_0.$$

Let $r_j = \sum_{i=j}^{d-t} (-1)^{i-j} \operatorname{rank} G_i = \sum_{i=t}^{d-j} (-1)^{d-j-i} \operatorname{rank} F_i$ and let I_j be the ideal generated by the r_j-minors of g_j for $j = 1, \ldots, d-t$. We have $(I_j)_p = A_p$ for every prime ideal $p \neq m$ because $G. \otimes A_p$ is exact and split. Therefore I_j is m-primary and $\dim A/I_j = 0 \leq d - t - j$ for $j = 1, \ldots, d-t$. We first consider the case where $t < d - 1$. By Theorem 1, we have $r_1 \geq t$. Let $Z = \operatorname{Ker} f_{d-1}$ and $B = \operatorname{Im} f_d$. Take any associated prime ideal p of A. Then $\dim A/p = d$. Since the complex $(0 \to Z \to F_d \to \cdots \to F_t \to 0) \otimes A_p$ is exact and split, Z_p is free and $\operatorname{rank} Z_p = \operatorname{rank} F_d - r_1$. As $Z_p/B_p \cong H_d(F.)_p \cong E(A_p/pA_p)$, we have $Z_p \neq 0$ and $\operatorname{rank} F_d \geq r_1 + 1$. Suppose $\operatorname{rank} F_d = r_1 + 1$. Then $Z_p \cong A_p$. Since $\lambda(Z_p) - \lambda(B_p) = \lambda(E(A_p/pA_p)) = \lambda(A_p)$, we have $B_p = 0$. Therefore we have $B = 0$ as B is a submodule of a free module. Then the complex $\cdots \to F_{d+2} \to F_{d+1} \to 0$ is exact, split and minimal, hence $F_i = 0$ for $i > d$. Therefore $\mu^i(A) = \operatorname{rank} F_i = 0$ for $i > d$, which means that A is Gorenstein. This is a contradiction. Hence we have $r(A) = \operatorname{rank} F_d > r_1 + 1 \geq t + 1 \geq n$, a contradiction. We now consider the case where $t = d - 1$. The ideal I_1 is an m-primary ideal generated by the maximal minors of f_{d-1}. Therefore $n \leq d = \operatorname{ht} I_1 \leq \operatorname{rank} F_d - \operatorname{rank} F_{d-1} + 1 \leq n$. Hence we have $d = n$, $\operatorname{rank} F_d = n$ and $\operatorname{rank} F_{d-1} = 1$. So there exist elements $x_1, \ldots, x_d$ in m such that $H_{d-1}(F.) \cong A/(x_1, \ldots, x_d)$. Since $H_{d-1}(F.)$ is of finite length, $x_1, \ldots, x_d$ is a system of parameters of A. As $H_m^{d-1}(A) \cong \operatorname{Hom}_A(H_{d-1}(D.), E(A/m)) \cong \operatorname{Hom}_A(H_{d-1}(F.), E(A/m))$, we have $(x_1, \ldots, x_d)H_m^{d-1}(A) = 0$. It is easy to see that $(x_1, \ldots, x_d)H_m^i(A/(x_1, \ldots, x_j)) = 0$ for $i + j < d$ because A is (S_{d-1}). By [11, (2.5), (2.1) and (1.5)], $\lambda(A/(x_1, \ldots, x_d)) - e(x_1, \ldots, x_d; A) = \lambda(H_m^{d-1}(A)) = \lambda(A/(x_1, \ldots, x_d))$, where $e(x_1, \ldots, x_d; A)$ denotes the multiplicity of A with respect to $x_1, \ldots, x_d$. Hence we have $e(x_1, \ldots, x_d; A) = 0$, which is a contradiction. Now the proof is completed.

By a similar argument we have the following theorem for modules.

Theorem 3. *Let M be a finitely generated A-module and let n be a positive integer. If $r(M) \leq n$ and $\hat{M}$ is (S_n) and equidimensional, then M is Cohen-Macaulay.*

Remark. The case where $n = 1$ in Theorem 3 is a special case of a conjecture of Foxby (cf. [4. Proposition (3.1)]).

Conjecture B in [4]. If $r(M) = 1$, then both M and $B = A/\operatorname{ann}(M)$ are Cohen-Macaulay and M is a dualizing module of B.

It is known that this conjecture is true in general as well as in the ring case ([10] and [4]. cf. [9] and [8, p. 66]).

In the proof of Theorem 2, we have $r_1 \geq t + 1$ in the case where $t < d - 1$ by [2, Theorem 3] if A contains a field. Therefore we have the following result.

Theorem 4. *Let n be a positive integer and assume that A contains a field.*

(1) *Suppose that A satisfies the following conditions*: (i) $r(A) \leq n$, (ii) $\hat{A}$ *is* (S_{n-2}), (iii) $\hat{A}$ *is strictly equidimensional* (*if* $n \leq 3$), *and* (iv) $\hat{A}_p$ *is Cohen-Macaulay for any p in* $\operatorname{spec}(\hat{A})$ *such that* $\dim \hat{A}_p < n$. *Then A is Cohen-Macaulay.*

(2)([5, Theorem (3.1) ii)]) *Let M be a finitely generated A-module. Suppose that M satisfies the following conditions*: (i) $r(M) \leq n$, (ii) $\hat{M}$ *is* (S_{n-1}), (iii) $\hat{M}$ *is equidimensional, and* (iv) $\hat{M}_p$ *is Cohen-Macaulay for any p in* $\operatorname{Supp}(\hat{M})$ *such that* $\dim \hat{M}_p \leq n$. *Then M is Cohen-Macaulay.*

Acknowledgement. The author wishes to express his thanks to Mr. Takeshi Kawasaki for sending him a preprint and to Prof. Shiro Goto for kind advice.

This work is supported by Grant-in-Aid for Scientific Research C-05640040 and Co-operative Research A-04302003 from the Ministry of Education of Japan.

References

[1] H. Bass: On the ubiquity of Gorenstein rings. Math. Z., **82**, 8–28 (1963).

[2] W. Bruns: The Evans-Griffith syzygy theorem and Bass numbers. Proc. Amer. Math. Soc., **115**, 939–946 (1992).

[3] D. Costa, C. Huneke and M. Miller: Complete local domains of type two are Cohen-Macaulay. Bull. London Math. Soc., **17**, 29–31 (1985).

[4] H.-B. Foxby: On the μ^i in a minimal injective resolution II. Math. Scand., **41**, 19-44 (1977).

[5] T. Kawasaki: Local rings of relatively small type are Cohen-Macaulay. Proc. Amer. Math. Soc. (to appear).

[6] T. Marley: Unmixed local rings of type two are Cohen-Macaulay. Bull. London Math. Soc., **23**, 43–45 (1991).

[7] T. Ogoma: Existence of dualizing complexes. J. Math. Kyoto Univ., **24**, 27–48 (1984).

[8] P. Roberts: Homological invariants of modules over commutative rings. Sém. Math. Sup., Univ. Montréal (1980).

[9] ——: Rings of type 1 are Gorenstein. Bull. London Math. Soc., **15**, 48–50 (1983).

[10] ——: Le théorème d'intersection. C. R. Acad. Sc. Paris, **304**, Sér, I, no. 7, pp. 177–180 (1987).

[11] N. V. Trung: Toward a theory of generalized Cohen-Macaulay modules. Nagoya Math. J., **102**, 1–49 (1986).

[12] W. V. Vasconcelos: Divisor theory in module category. Math. Studies, **14**, North Holland (1975).