18. *Equidimensional Toric Extensions of Symplectic Groups*

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§ 0. G (resp. T) will always stand for a connected reductive complex algebraic group (resp. connected complex algebraic torus). We will use any of the notations ρ, (ρ, G) or (V, G) to denote a finite dimensional representation $\rho : G \to GL(V)$ over the complex number field $\boldsymbol{C}$ and often confuse ρ with the affine space V. An algebraic action of G on an affine variety X (abbr. (X, G)) is said to be *cofree* (resp. *equidimensional*), if $\boldsymbol{C}[X]$ is $\boldsymbol{C}[X]^G$-free (resp. if $X \to X/G$ is equidimensional), where $\boldsymbol{C}[X]$ denotes the affine coordinate ring of X and X/G denotes the algebraic quotient of X. On the other hand, (X, G) is said to be *stable*, if X contains a non-empty open subset consisting of closed G-orbits. For toric actions, we have proved in [5] the following result, which is fundamental in this paper:

Theorem 0.1 ([5]). *Let X be an affine conical factorial variety with an algebraic action of T compatible with the conical structure of X. Let W be a dual of a homogeneous T-submodule of $\boldsymbol{C}[X]$ which minimally generates $\boldsymbol{C}[X]$ as a $\boldsymbol{C}$-algebra. Then (X, T) is stable and equidimensional if and only if so is (W, T). If these conditions are satisfied, then both actions (X, T) and (W, T) are cofree.*

V. L. Popov and V. G. Kac conjectured that equidimensional representations are cofree. Concerning their conjecture, we will obtain

Theorem 0.2. *Suppose that the commutator subgroup of G is symplectic and of rank ≥ 3. Then finite dimensional equidimensional stable representations of G are cofree.*

We denote by G' the commutator subgroup of G and say that (V, G) is *relatively equidimensional* (resp. *relatively stable*), if $(V/G', G/G')$ is equidimensional (resp. stable). The purpose of this paper is to show

Theorem 0.3. *Under the same circumstances as in* (0.2), *suppose that the natural action of $Z(G)^0$ on $V/V^{G'}$ is nontrivial. If (V, G) is relatively stable and relatively equidimensional, then the restriction of (V, G) to G' (i.e., $((V, G), G')$) is cofree.*

This assertion does not hold, in the case where the semisimple rank of G is ≤ 2 (cf. [4]). Since equidimensional (resp. stable) representations are relatively equidimensional (resp. relatively stable), (0.2) follows from this and the classification [1] obtained by O. M. Adamovich and G. W. Schwarz. Some (calculative) parts of our proofs are left to the readers. The related study on

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other simple groups shall be published in a forthcoming paper.

§1. Let $\mathfrak{X}(T)$ denote the rational linear character group of T over $\boldsymbol{C}$ and we regard this group as an additive group. A sequence $(\chi_1, \ldots, \chi_m)$ in $\mathfrak{X}(T)$ is said to be *uniquely and positively related* (abbr. *UPR*), if $\mathrm{rk}\langle\chi_1, \ldots, \chi_m\rangle = m-1$ and $\sum_{i=1}^m a_i\chi_i = 0$ for some $0 < a_i \in \boldsymbol{Q}$. In general a sequence $(\psi_1, \ldots, \psi_s)$ in $\mathfrak{X}(T)$ is said to be *stably equidimensional* (abbr. *SEQ*), if $\sum_{i=1}^u \langle\psi_{\sigma(s_{i-1}+1)}, \ldots, \psi_{\sigma(s_i)}\rangle = \bigoplus_{i=1}^u \langle\psi_{\sigma(s_{i-1}+1)}, \ldots, \psi_{\sigma(s_i)}\rangle$ and $(\psi_{\sigma(s_{i-1}+1)}, \ldots, \psi_{\sigma(s_i)})$ $(1 \leq i \leq u)$ are UPR, for some permutation σ of $\{1, \ldots, s\}$, $0 \leq u \in \boldsymbol{Z}$ and $0 \leq s_i \in \boldsymbol{Z}$ $(1 \leq i \leq u)$ such that $0 = s_0 < s_1 < \cdots < s_u = m$. For any $a_i \in \boldsymbol{N}$, $(a_1\psi_1, \ldots, a_s\psi_s)$ is SEQ if and only if so is $(\psi_1, \ldots, \psi_s)$.

Lemma 1.1. *Let δ_i $(0 \leq i \leq 3)$ be linear characters of T.*

(1) *If $(\delta_1, \delta_2, \delta_1+\delta_2)$ is a subsequence of a SEQ sequence, then $\delta_1+\delta_2 = 0$.*

(2) *Suppose that $\delta_0 \neq 0$. Then $(\delta_0, \delta_1+\delta_2, \delta_0+\delta_1, 2\delta_0+\delta_1+\delta_2, \delta_0+\delta_2)$ is a subsequence of a SEQ sequence if and only if $-\delta_0 = \delta_1 = \delta_2$ or $2\delta_0+\delta_1+\delta_2 = 0$ and $\mathrm{rk}\langle\delta_1, \delta_2\rangle = 2$.*

(3) *Let $(a_{ij}) \in GL_2(\boldsymbol{Q})$ such that $a_{11}, a_{21} \geq 0$. If $(\delta_1, \delta_2, a_{11}\delta_1+a_{12}\delta_2, a_{21}\delta_1+a_{22}\delta_2)$ is a subsequence of a SEQ sequence, then $\delta_1 = \delta_2 = 0$.*

(4) *Suppose that $\delta_i \neq 0$ $(0 \leq i \leq 3)$. Then $(\delta_0+\delta_1, \delta_0+\delta_2, \delta_0+\delta_3, \delta_1+\delta_2, \delta_1+\delta_3, \delta_2+\delta_3)$ is a subsequence of a SEQ sequence if and only if $\mathrm{rk}\langle\delta_1, \delta_2, \delta_3\rangle = 3$ and $\delta_0 = \delta_1+\delta_2+\delta_3$ or, up to a replacement of indices of δ_i, $\mathrm{rk}\langle\delta_0, \delta_1\rangle = 2$, $\delta_2 = -\delta_0$ and $\delta_3 = -\delta_1$.*

We may assume that $G = G' \times T$ and T acts faithfully on the representation space V of G. A representation $(V, G' \times T)$ is said to be *irredundant along trivial parts* (resp. *relatively irredundant along trivial parts*), if T acts nontrivially on V (resp. V/G') and, for any nonzero subspace U of $V^{G'}$, $T \neq (\cap_{y \in V/U} T_y) \times (\cap_{y \in U} T_y)$ (resp. $T|_{V/G'} \neq (\cap_{z \in (V/U)/G'} (T|_{V/G'})_z) \times (\cap_{z \in U} (T|_{V/G'})_z)$).

Lemma 1.2. *Suppose that* $\mathrm{Ker}(T|_{V/V^{G'}} \to \mathrm{Aut}((V/V^{G'})/G'))^0$ *is trivial. Then (V, G) is irredundant along trivial parts if and only if it is relatively irredundant along trivial parts.*

Since V/G' is a conical factorial variety with an action of the torus $G/G' \cong T$, by (0.1), we have the following two results:

Proposition 1.3. *Let $\chi_i \in \mathfrak{X}(T)$ to satisfy $(V^{G'}, T) = \chi_1 \oplus \cdots \oplus \chi_s$ for some $0 \leq s \in \boldsymbol{Z}$ and let $\psi_i \in \mathfrak{X}(T)$ such that $\psi_1 \oplus \cdots \oplus \psi_m$ is isomorphic to a homogeneous T-submodule minimally generating $\boldsymbol{C}[V/V^{G'}]^{G'}$ for some $0 \leq m \in \boldsymbol{Z}$. Then (V, G) is relatively equidimensional, relatively stable and relatively irredundant along trivial parts if and only if $\{\psi_i \mid \psi_i \neq 0, 1 \leq i \leq m\} \neq \emptyset$, $(\psi_1, \ldots, \psi_m, \chi_1, \ldots, \chi_s)$ is SEQ and any nontrivial subsequence of $(\chi_1, \ldots, \chi_s)$ is not UPR.*

Lemma 1.4. *Let $\varrho \subseteq V/V^T$ be an irreducible subrepresentation of G. If (V, G) is relatively stable and relatively equidimensional, then*

$$\mathrm{emb}(\boldsymbol{C}[V^T \oplus \varrho]^{G'}) \leq \mathrm{emb}(\boldsymbol{C}[V^T]^{G'}) + 1.$$

Lemma 1.5. *Let ϱ be a nontrivial irreducible representation and φ a repre-*

sentation of G'. *Suppose that* $\mathrm{emb}(C[\varphi \oplus \varrho]^{G'}) \leq \mathrm{emb}(C[\varphi]^{G'}) + 1$.

(1) $\dim C[\varrho]^H \leq 1$ *for an isotropy subgroup* H *of* G' *at a* G'*-semisimple point of* φ.

(2) *Suppose that* G' *is simple and* φ *is irreducible (may be trivial). Then* $(\varphi \oplus \varrho, G')$ *is coregular.*

Proof. From the inequality and the slice étale theorem, we infer that $\dim(C[\varphi_x \oplus \varrho]^H) = \dim(C[\varphi \oplus \varrho]^{G'}) \leq \dim(C[\varphi]^{G'}) + 1 = \dim(C[\varphi_x]^H) + 1$, where $x \in \varphi$ such that $G'x$ is a closed orbit with $G_x = H$ and (φ_x, H) denotes the slice representation of (φ, G') at x. Thus (1) follows. For (2), applying (1) and Popov's criterion on stability, we see that an isotropy group at a general position of (φ, G') is of dimension ≥ 1. Since G' is simple, φ is coregular (e.g. [3, 2, 6]) and so we get the assertion. Q.E.D.

§2. *We suppose that* G' *is a connected simply-connected simple algebraic group of type* $\mathbf{C}_n (n \geq 3)$. Let Φ_1 be the natural representation of G' of degree $2n$ and Φ_i, $i \leq n$, denote the highest irreducible representation of the i-th exterior power of Φ_1. The tensor product of representations (ρ, G') and $\chi \in \mathfrak{X}(T)$ is denoted by $(\rho \cdot \chi, G' \times T)$ and both ρ and χ are naturally regarded as representations of G.

Proposition 2.1. *Suppose that* $n = 3$ *and* V *contains* Φ_3 *as a* G'*-module. Then* $(V, G' \times T)$ *is relatively irredundant along trivial parts, relatively stable and relatively equidimensional if and only if it is equivalent to one of the representations listed in Table I.*

Table I $((V, \mathbf{C}_3 \times T), \mathbf{C}_3) \supseteq \Phi_3$

	$\mathbf{C}_3 \times T$	condition
1	$\Phi_3 \oplus \Phi_1 \cdot \psi \oplus \Phi_1 \cdot (-\psi)$	
2	$\Phi_3 \oplus \Phi_1 \cdot \psi \oplus \chi_1 \oplus \cdots \oplus \chi_s$	$(2\psi, \chi_1, \ldots, \chi_s)$ UPR
3	$\Phi_3 \cdot \delta \oplus 2\Phi_1 \cdot (-\delta)$	
4	$\Phi_3 \cdot \delta \oplus \Phi_1 \cdot \psi_1 \oplus \Phi_1 \cdot \psi_2$	$\mathrm{rk}\langle \psi_1, \psi_2 \rangle = 2$, $2\delta = -\psi_1 - \psi_2$
5	$\Phi_3 \cdot \delta \oplus \Phi_1 \cdot \psi$	$\delta = a \cdot \psi$, $-1 > a \in \boldsymbol{Q}$
6	$\Phi_3 \cdot \delta \oplus \Phi_1 \cdot (-\delta) \oplus \chi_1 \oplus \cdots \oplus \chi_s$	$(\delta, \chi_1, \ldots, \chi_s)$ UPR
7	$\Phi_3 \cdot \delta \oplus \Phi_1 \cdot \psi \oplus \chi_1 \oplus \cdots \oplus \chi_s$	$(\delta, \delta + \psi, \chi_1, \ldots, \chi_s)$ UPR
8	$\Phi_3 \cdot \delta \oplus \chi_1 \oplus \cdots \oplus \chi_s$	$(\delta, \chi_1, \ldots, \chi_s)$ UPR

Comment: $0 \neq \delta, \psi, \psi_i, \chi_i \in \mathfrak{X}(T)$; $s \geq 1$

Proposition 2.2. *Let* $0 \leq u \in \boldsymbol{Z}$. *Let* ψ_i $(1 \leq i \leq m, 0 \leq m \in \boldsymbol{Z})$ *and* χ_j $(1 \leq j \leq s, 0 \leq s \in \boldsymbol{Z})$ *be nonzero linear characters of* T. *Then a representation* $(V, G' \times T) = u\Phi_1 \oplus \Phi_1 \cdot \psi_1 \oplus \cdots \oplus \Phi_1 \cdot \psi_m \oplus \chi_1 \oplus \cdots \oplus \chi_s$ *is relatively irredundant along trivial parts, relatively stable and relatively equidimensional if and only if one of the conditions listed in Table II holds.*

Theorem 2.3. $(V, G' \times T)$ *is relatively irredundant along trivial parts, relatively stable and relatively equidimensional if and only if it is equivalent to one of the representations listed in Tables I and II.*

Theorem 2.4. *Suppose that a representation* $(V, G' \times T)$ *is irredundant along trivial parts. Then* $(V, G' \times T)$ *is relatively stable and relatively equidimensional if and only if it is equivalent to one of the representations listed* *in Tables I-III.*

In this theorem, we can drop the assumption on "irredundancy", although the condition on linear characters may be more complicated.

Table II $u\Phi_1 \oplus \Phi_1 \cdot \psi_1 \oplus \cdots \oplus \Phi_1 \cdot \psi_m \oplus \chi_1 \oplus \cdots \oplus \chi_s$

	u	m	s	character relation
1	1	1	≥ 1	$(\psi_1, \chi_1, \ldots, \chi_s)$ UPR
2	0	2	≥ 1	$(\psi_1 + \psi_2, \chi_1, \ldots, \chi_s)$ UPR
3	0	3	≥ 1	$(\psi_1 + \psi_2, \psi_1 + \psi_3, \psi_2 + \psi_3, \chi_1, \ldots, \chi_s)$ UPR
4	0	3	≥ 1	$\psi_2 = -\psi_1$, $(\psi_1 + \psi_3, \psi_2 + \psi_3, \chi_1, \ldots, \chi_s)$ UPR
5	0	3	≥ 1	$\psi_1 + \psi_2 \in \boldsymbol{Q}_- \cdot (\psi_1 + \psi_3)$, $(\psi_1 + \psi_3, \psi_2 + \psi_3, \chi_1, \ldots, \chi_s)$ UPR
6	0	3	0	$(\psi_1 + \psi_2, \psi_1 + \psi_3, \psi_2 + \psi_3)$ UPR
7	1	2	0	$\psi_2 = -\psi_1$
8	0	4	0	$\mathrm{rk}\langle\psi_1, \psi_2, \psi_3\rangle = 3$, $\psi_4 = -\psi_1 - \psi_2 - \psi_3$
9	0	4	0	$\mathrm{rk}\langle\psi_1, \psi_2\rangle = 2$, $\psi_3 = -\psi_1$, $\psi_4 = -\psi_2$
10	0	3	0	$\psi_2 = -\psi_1$, $\psi_1 + \psi_3 \in \boldsymbol{Q}_- \cdot (\psi_3 - \psi_1)$

Comment: $0 \neq \psi_i, \chi_j \in \mathfrak{X}(T)$; $\boldsymbol{Q}_- = \{r \in \boldsymbol{Q} \mid r < 0\}$;
up to a replacement of indices of ψ_i's ("4, 5, 9, 10")

Table III $\mathrm{Ker}(T|_{V/V^{\mathbf{C}_n}} \to \mathrm{Aut}((V/V^{\mathbf{C}_n})/\mathbf{C}_n))^0 \neq \{1\}$

	$\mathbf{C}_n \times T$	condition
1	$\Phi_1 \cdot \psi \oplus \chi_1 \oplus \cdots \oplus \chi_s$	$(\chi_1, \ldots, \chi_s)$ SEQ
2	$\Phi_1 \cdot \psi_1 \oplus \Phi_1 \cdot \psi_2 \oplus \chi_1 \oplus \cdots \oplus \chi_s$	$(\psi_1 + \psi_2, \chi_1, \ldots, \chi_s)$ SEQ
3	$\Phi_1 \cdot \psi \oplus \Phi_1 \cdot (-\psi) \oplus \chi_1 \oplus \cdots \oplus \chi_s$	$(\chi_1, \ldots, \chi_s)$ SEQ
4	$\Phi_1 \cdot \psi \oplus \Phi_2 \oplus \chi_1 \oplus \cdots \oplus \chi_s$	$(\chi_1, \ldots, \chi_s)$ SEQ
5	$\Phi_2 \oplus \Phi_1 \cdot \psi \oplus \Phi_1 \cdot (-\psi) \oplus \chi_1 \oplus \cdots \oplus \chi_s$	$(\chi_1, \ldots, \chi_s)$ SEQ

Comment: $0 \neq \psi, \psi_i, \chi_j \in \mathfrak{X}(T)$; $\mathrm{rk}\langle\psi_1, \psi_2\rangle = 2$; $s \geq 0$

§ 3. This section is devoted to the proof of the results in §2. By (1.5.2) and [2, 6], we get

Lemma 3.1. *Let ϱ be a nontrivial irreducible representation of G' and φ a representation of G' without nonzero trivial subrepresentations. Then $\boldsymbol{C}[\varphi \oplus \varrho]^{G'} = \boldsymbol{C}[\varphi]^{G'}$ if and only if $\varrho = \Phi_1$ and $\varphi = 0$ or Φ_2.*

Proposition 3.2. *Suppose that T is nontrivial. Then the natural action $(V/G', T)$ is trivial if and only if $(V/V^G, G)$ is equivalent to one of the representations listed in Table III deleting "2" with the extra condition that $s = 0$.*

Proof. We see that $V^{G'} = 0$ and may express (V, G) as $\varphi_1 \oplus \cdots \oplus \varphi_u \oplus \varrho_1 \cdot \psi_1 \oplus \cdots \oplus \varrho_m \cdot \psi_m$ for some nontrivial irreducible representations φ_i, ϱ_j of G', nonzero $\psi_j \in \mathfrak{X}(T)$, $0 \leq u \in \boldsymbol{Z}$ and $m \in \boldsymbol{N}$. Since $(V/G', T)$ is trivial, from (3.1), one infers that $u = 1$ and $\varphi_1 \cong \Phi_2$ if $u > 0$ and that $\varrho_j \cong \Phi_1$. By the first main theorem in invariant theory due to H. Weyl, we see that $\boldsymbol{C}[\oplus_{j=1}^m \Phi_1]^{G'}$ is minimally generated by nonzero homogeneous invariants in the subrepresentations which are isomorphic to $\Phi_1 \otimes \Phi_1 \subseteq \boldsymbol{C}[\Phi_1 \oplus \Phi_1]$. By (1.1.1), we see that $\psi_j + \psi_k = 0$ for any $j \neq k$. Thus $m \leq 2$, and hence $\boldsymbol{C}[V]^{G'}$ is known. The remainder of the assertion follows from the datum on their fundamental invariants. Q.E.D.

For a representation (V, G) in Tables I-III, by [2, 6, 7], we see that $((V, G), G')$ is cofree, and $\boldsymbol{C}[V]^{G'}$ can be determined. We can show that it is relatively equidimensional and relatively stable, and from (1.3) we derive that (V, G) listed in Tables I and II is relatively irredundant along trivial parts. The "only if" part of (2.4) follows from (0.1), (1.2), (2.3) and (3.2), because, for any (V, G) in (3.2), $\boldsymbol{C}[V]^{G'}$ is a polynomial ring. *Hereafter we assume that (V, G) is irredundant along trivial parts, relatively stable and relatively equidimensional.* Note $(V/V^T)^{G'} \neq V/V^T$ (and $(V/V^{G'})^T \neq V/V^{G'}$).

Lemma 3.3. *Any nontrivial irreducible subrepresentation of (V^T, G') is equivalent to one of Φ_1, Φ_2 and (Φ_3, C_3). Conversely if $(V^T, G') \supseteq a\Phi_2 \oplus b\Phi_3$ $(0 \leq a, b \in \boldsymbol{Z})$, then $a + b \leq 1$.*

Proof. Since unimodular toric actions are stable, from (1.4) and (1.5.1), we infer that any nontrivial subrepresentation of (V^T, G') does not have a principal closed isotropy subgroup whose identity component is a torus. Thus the first assertion follows from [3]. By the additivity of indices, the second assertion is also a consequence of the above remark, because indices of $\Phi_2 \oplus \Phi_3$, $2\Phi_3$, $2\Phi_2$ $(n \geq 4)$ are strictly greater than 1 (cf. [3]) and the identity component of a principal closed isotropy subgroup of $(2\Phi_2, \mathrm{C}_3)$ is a torus of rank one. Q.E.D.

Lemma 3.4. *Any nontrivial irreducible subrepresentation ϱ of $(V/V^T, G')$ is equivalent to Φ_1 or (Φ_3, C_3). Conversely if $(V/V^T, G') \supset \Phi_3$, then $((V, G), G') \not\supseteq 2\Phi_3$ and $(V/V^T, G') \cong \Phi_3 \oplus d\Phi_1$ for some $0 \leq d \in \boldsymbol{Z}$.*

Proof. The first assertion follows from the inequality $\dim (\boldsymbol{C}[\varrho]^{G'}) \leq 1$ (cf. (1.5.1)) and [2, 6]. Since $\boldsymbol{C}[2\Phi_3]^{\mathrm{C}_3}$ is generated by $\boldsymbol{Z}^2$-homogeneous polynomials of degrees (4,0), (3,1), (1,3) and (0,4), by (1.1.3), we see that $((V, G), G') \not\supseteq 2\Phi_3$. Q.E.D.

Proof of (2.1). Suppose that (V, G) is relatively irredundant along trivial parts. First, we show the assertion in the case where $V \supseteq \Phi_3 \cdot \delta$ for a nonzero $\delta \in \mathfrak{X}(T)$. Then, since $(\Phi_2 \oplus \Phi_3, G')$ is not coregular (cf. [2, 6]), by (1.4), (1.5.2) and the second assertion of (3.4), we see that $((V, G), G') = \Phi_3 \oplus d\Phi_1$ for some $0 \leq d \in \boldsymbol{Z}$. $\boldsymbol{C}[\Phi_3 \oplus \Phi_1 \oplus \Phi_1]^{G'}$ is generated by $\boldsymbol{Z}^3$-homogeneous polynomials of degrees (4,0,0), (0,1,1), (2,2,0), (2,1,1) and (2,0,2). Suppose $((V, G), G') \supseteq \Phi_1 \cdot \psi_1 \oplus \Phi_1 \cdot \psi_2 \oplus \Phi_1 \cdot \psi_3$ for $\psi_i \in \mathfrak{X}(T)$. By (1.1.2.), we see that $2\delta = -\psi_1 - \psi_2 = -\psi_1 - \psi_3 = -\psi_2 - \psi_3$ and so $-\delta = \psi_1 = \psi_2 = \psi_3 = 0$, because $(\Phi_1 \otimes \Phi_1)^{G'} \neq 0$. This is a contradiction and so $d \leq 2$. If $d = 2$, then, by (1.1.2), we see that (V, G) is equivalent to "3" or "4" in Table I. For $d \leq 1$, the assertion follows from (1.3).

Next suppose that $((V/V^T, G), G') \not\supseteq \Phi_3$. Then, by the second assertion of (3.4), we see that $((V/V^T)/(V/V^T)^{G'}, G') = d\Phi_1$ for some $d \in \boldsymbol{N}$. If $V^T \supseteq \Phi_3 \oplus \Phi_1$, then $\mathrm{emb}\,(\boldsymbol{C}[V^T \oplus \Phi_1]^{G'}) \geq \mathrm{emb}(\boldsymbol{C}[V^T]^{G'}) + 3$. Thus, by (1.4) and (3.3), we have $V^T = \Phi_3$. Suppose that $V/V^T \supseteq \Phi_1 \cdot \psi_1 \oplus \Phi_1 \cdot \psi_2 \oplus \Phi_1 \cdot \psi_3$ for $\psi_i \in \mathfrak{X}(T)$. Then, applying (1.1.1) to the subalgebras isomorphic to $\boldsymbol{C}[\Phi_3 \oplus \Phi_1 \oplus \Phi_1]^{G'}$, we see that $\psi_1 + \psi_2 = \psi_2 + \psi_3 = \psi_3 + \psi_1 = 0$, which implies $\psi_i = 0$. Thus $d = 1$ or 2 and, especially in case of $d = 2$, $V = \Phi_3 \oplus \Phi_1 \cdot \psi \oplus \Phi_1 \cdot (-\psi) \oplus \chi_1 \oplus \cdots \oplus \chi_s$ for some nonzero $\psi, \chi_j \in$

$\mathfrak{X}(T)$. The remainder of the assertion follows from (1.3). Q.E.D.

Proof of (2.2). Suppose that (V, G) is relatively irredundant along trivial parts. Clearly $m > 0$. As in the proof of (3.2), we see that $u \leq 1$ and, moreover, that $\psi_i + \psi_j = 0$ $(i \neq j)$ unless $u = 0$. Thus, in the case where $u > 0$, we infer that $u = 1$ and $m = 1$ or 2, and, by (1.3), that (V, G) is equivalent to "1" or "7" in Table II.

Next we treat the case where $u = 0$. Assume that $m \leq 5$. Then, by (1.3), the equivalent conditions of (1.1.4) are satisfied for $(\delta_0, \delta_1, \delta_2, \delta_3) = (\psi_4, \psi_1, \psi_2, \psi_3)$ and $(\psi_5, \psi_1, \psi_2, \psi_3)$ respectively. Suppose that rk $\langle\psi_1, \psi_2, \psi_3\rangle = 3$. Then $\psi_1 + \psi_2 + \psi_3 + \psi_4 = \psi_1 + \psi_2 + \psi_3 + \psi_5 = 0$, which implies $\psi_4 = \psi_5$. Thus there are two distinct homogeneous semiinvariants of G relative to $\psi_1 + \psi_4 = \psi_1 + \psi_5$ in a minimal generating system of $\boldsymbol{C}[V]^{G'}$, which contradicts (1.3). We may assume that rk $\langle\psi_4, \psi_1\rangle = 2$, $\psi_2 = -\psi_4$ and $\psi_3 = -\psi_1$. Since rk $\langle \psi_1, \psi_2, \psi_3\rangle = 2$, we see that $\psi_5 = -\psi_1$, $-\psi_2$ or $-\psi_3$. Say $\psi_5 = -\psi_1$. Then $\psi_5 = \psi_3$, and hence, as in the above case, we similarly get a contradicition. Consequently $m \leq 4$, and $((V, G), G')$ is cofree (cf. [7]). Using (1.3), we can show the remainder of the assertion. Q.E.D.

Lemma 3.5. *Suppose* $((V, G), G') \supseteq \Phi_2$. *Then* (V, G) *is equivalent to* "4" *or* "5" *in Table III and is relatively redundant along trivial parts.*

Proof. From (3.3), one sees that $V^T = \Phi_2 \oplus d\Phi_1$ and, from (2.1), that $(V/V^T, G') = c\Phi_1$, for some $0 \leq d \in \boldsymbol{Z}$ and $c \in \boldsymbol{N}$. Since $\Phi_1 \otimes \Phi_1 \supseteq \Phi_2 \cong \Phi_2^*$ and the free $\boldsymbol{C}[\Phi_2]^{G'}$-module $\boldsymbol{C}[\Phi_2]$ is of rank $n-1$ (cf. [7]), by (1.4), we deduce that $d = 0$ and, by (0.1), see that $\psi_1 + \psi_2 = 0$, if $V \supseteq \Phi_1 \cdot \psi_1 \oplus \Phi_1 \cdot \psi_2$ for nonzero $\psi_i \in \mathfrak{X}(T)$. This implies that (V, G) is equivalent to $\Phi_2 \oplus \Phi_1 \cdot \psi \oplus \Phi_1 \cdot (-\psi) \oplus \chi_1 \oplus \cdots \oplus \chi_s$ or $\Phi_2 \oplus \Phi_1 \cdot \psi \oplus \chi_1 \oplus \cdots \oplus \chi_s$ for some $0 \leq s \in \boldsymbol{Z}$ and nonzero ψ, $\chi_i \in \mathfrak{X}(T)$. In both cases, T acts trivially on $\boldsymbol{C}[V/V^{G'}]^{G'}$, and, by (0.1), we see that $(\chi_1, \ldots, \chi_s)$ is SEQ. Q.E.D.

By (3.5) and the first assertions of (3.3) and (3.4), we see that (2.3) is a consequence of (2.1) and (2.2). The main result (0.3) follows from (2.4) and [7].

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