17. *Coefficient Bounds for the Inverse of a Function whose Derivative has a Positive Real Part*

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Abstract: In this paper we study the coefficient bounds for the inverse of a function whose derivative has a positive real part. We prove the conjecture posed by R. J. Libera and E. J. Złotkiewicz [3].

1. Introduction and conclusion. Let S denote the class of functions of the form

(1) $$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

which are analytic and univalent in $\Delta = \{z : |z| < 1\}$. De Branges [1] has proved that $a_n (n = 2,3,\cdots)$ are bounded by those of the Koebe function, $k(z) = z + 2z^2 + 3z^3 + \cdots$, that is, $|a_n| \leqslant n \ (n \geqslant 2)$.

The inverse of $f(z)$ has a series expansion in some disk about the origin of the form

(2) $$F(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots.$$

It was shown early (see [2]) that the inverse of the Koebe function provides the best bound for all $|\gamma_k|$.

As is usually the case, we let $\mathscr{P}$ be the family of functions

(3) $$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

regular and with $\operatorname{Re} p(z) > 0 (z \in \Delta)$. Furthermore we denote by J the class of all functions of form (1) which satisfies

(4) $$\operatorname{Re} f'(z) > 0, \ z \in \Delta.$$

This is the family studied widely. Let the inverse of $f(z)$ belonging to J have the form (2). R. J. Libera and E. J. Złotkiewicz [3] found sharp bounds for the first six coefficients of $F(w)$; the extremal function is $\tilde{F}_0(w)$ which corresponds to $\tilde{f}_0(z) = -z - 2\log(1 - z)$. They also conjectured that $\tilde{F}_0(w)$ gives the sharp upper bounds for other (perhaps even all) coefficients. In this paper we prove this conjecture, and the method is very succinct. Our conclusion is

Theorem. Let $f(z)$ be in J and the inverse of $f(z)$ be $F(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$. Then

$$|\gamma_n| \leqslant B_n (n = 2,3,\cdots)$$

where $B_n (n = 2,3,\cdots)$ are the coefficients of $F_0(w)$ which corresponds to $f_0(z) = -z + 2\log(1 + z)$. The function attaining the equalities is the inverse of $f_0(z)$.

2. The proof of the theorem. It's easy to know that $1/p(z) \in \mathscr{P}$ when $p(z) \in \mathscr{P}$. So if $f(z) \in J$, then there exists a $p(z) \in \mathscr{P}$ such that

$$f'(z) = 1/p(z) \qquad (z \in \Delta).$$

Because $f'(z)F'(w) = 1$, we have

(5) $$1/F'(w) = 1/p(F(w)),$$

so

$$F'(w) - 1 = p(F(w)) - 1 = \sum_{n=1}^{\infty} C_n[F(w)]^n,$$

that is

(6) $$\sum_{n=1}^{\infty} (n+1)\gamma_{n+1} w^n = \sum_{n=1}^{\infty} C_n[F(w)]^n = \sum_{n=1}^{\infty} [\sum_{j=1}^{n} C_j K_{n-j}^{(j)}] w^n,$$

where $K_{n-j}^{(j)}$ is the coefficient of w^n in the series expansion of $[F(w)]^j$ ($j = 1,2,\cdots$), specially, $K_0^{(n)} = 1$ ($n = 1,2,\cdots$). It is obvious that

$$K_{n-j}^{(j)} = K_{n-j}^{(j)}(\gamma_2, \gamma_3, \ldots, \gamma_n) \quad (n \geqslant 2)$$

is the non-negative coefficient polynomial of γ_k ($k = 2,3,\ldots,n$), so

(7) $$|K_{n-j}^{(j)}| \leqslant K_{n-j}^{(j)}(|\gamma_2|, |\gamma_3|, \ldots, |\gamma_n|) \qquad (n \geqslant 2).$$

From (6) we have

(8) $$2\gamma_2 = c_1, \quad (n+1)\gamma_{n+1} = \sum_{j=1}^{n} c_j K_{n-j}^{(j)} \quad (n \geqslant 2),$$

thus

(9) $$\begin{cases} 2|\gamma_2| \leqslant 2, \\ (n+1)|\gamma_{n+1}| \leqslant \sum_{j=1}^{n} |c_j| \cdot |K_{n-j}^{(j)}| \end{cases}$$
$$\leqslant 2\sum_{j=1}^{n} K_{n-j}^{(j)}(|\gamma_2|, |\gamma_3|, \ldots, |\gamma_n|) \quad (n \geqslant 2),$$

where we have used (7) and the well-known results $|c_n| \leqslant 2$ ($n = 1,2,\cdots$).

On the other hand,

$$f_0'(z) = (1-z)/(1+z),$$

so

(10) $$1/F_0'(w) = (1 - F_0(w))/(1 + F_0(w)),$$

that is,

$$F_0'(w) = 1 + 2\sum_{n=1}^{\infty} [F_0(w)]^n.$$

Similarly to (8), we have

(11) $$2B_2 = 2, \quad (n+1)B_{n+1} = 2\sum_{j=1}^{n} H_{n-j}^{(j)} \quad (n \geqslant 2),$$

where $H_{n-j}^{(j)}$ is the coefficient of w^n in $[F_0(w)]^j$ ($j = 1,2,\cdots$), and

(12) $$H_{n-j}^{(j)} = K_{n-j}^{(j)}(B_2, B_3, \ldots, B_n) \quad (n \geqslant 2)$$

is the non-negative coefficient polynomial of B_k ($k = 2,3,\ldots,n$), $H_0^{(n)} = 1$ ($n = 1,2,\cdots$).

Next we prove that all B_n ($n = 2,3,\cdots$) are positive. From (10) we can also get

$$1 + F_0(w) = F_0'(w)(1 - F_0(w)).$$

Substituting the series expansion of $F_0(w)$ into this equality, and comparing the coefficients of both sides, we get

(13) $B_2 = B_1 = 1,$

$$(n+1)B_{n+1} = 2B_n + \sum_{j=1}^{n} (j+1)B_{j+1} \cdot B_{n-j} \ (n \geqslant 2).$$

For $B_1 = 1 > 0$ and $B_2 = 1 > 0$, from the recurrence formulas (13) we know all $B_n (n = 2,3,\cdots)$ are positive, so from (11) and (12) we obtain

(14)
$$(n+1)B_{n+1} = 2\sum_{j=1}^{n} H_{n-j}^{(j)} = 2\sum_{j=1}^{n} K_{n-j}^{(j)}(B_2, B_3, \ldots, B_n).$$

From the first inequality of (9) we can obtain

(15)
$$|\gamma_2| \leqslant 1 = B_2.$$

Therefore, from the second inequality of (9) and by induction we obtain

$$(n+1)|\gamma_{n+1}| \leqslant 2\sum_{j=1}^{n} K_{n-j}^{(j)} (|\gamma_2|, |\gamma_3|, \ldots, |\gamma_n|) \leqslant 2\sum_{j=1}^{n} K_{n-j}^{(j)} (B_2\ B_3, \ldots, B_n) = (n+1)B_{n+1} \quad (n \geqslant 2),$$

that is,

(16)
$$|\gamma_{n+1}| \leqslant B_{n+1} \quad (n \geqslant 2).$$

(15) and (16) are the inequalities we need to prove. It is obvious that $f_0(z) = -z + 2\log(1+z)$ belongs to J and the inverse of $f_0(z)$ attains the equalities. The proof of the theorem is completed.

References

[1] De Branges: A proof of the Bieberbach conjecture. Acta Math., **154**, 137–152 (1985).

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[3] R. J. Libera and E. J. Złotkiewicz: Coefficient bounds for the inverse of a function with derivative in $\mathscr{P}$. Proc. Amer. Math. Soc., **87**, 251–257 (1983).