75. *Hilbert Spaces of Analytic Functions Associated with Generating Functions of Spherical Functions on $U(n)/U(n-1)$*

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1. Introduction. Let $\boldsymbol{R}$ or $\boldsymbol{C}$ be the field of real or complex numbers, $S(\boldsymbol{R}^n)$ or $S(\boldsymbol{C}^n)$ the unit sphere in $\boldsymbol{R}^n$ or $\boldsymbol{C}^n$ and $x \mapsto \bar{x}$ the usual conjugation in $\boldsymbol{C}$.

We denote by F the Hilbert space of analytic functions $f(w)$ of n complex variables $w = {}^t(w_1, w_2, \ldots, w_n) \in \boldsymbol{C}^n$, with the inner product defined by

$$(f, g) = \pi^{-n} \int_{C^n} \overline{f(w)} g(w) \exp(-|w_1|^2 - \cdots - |w_n|^2) dw_1 \cdots dw_n,$$

where

$$dw_1 \cdots dw_n = du_1 \cdots du_n dv_1 \cdots dv_n,\ w_j = u_j + iv_j (u_j, v_j \in \boldsymbol{R}),$$

and by H the usual Hilbert space $L^2(\boldsymbol{R}^n)$.

V. Bargmann constructed in [1] a unitary mapping A from H onto F given by an integral operator whose kernel is considered as a generating function of the Hermite polynomials. More precisely, $f = A\phi$ for $\phi \in H$ is defined by

$$f(w) = \int_{R^n} A(w, t) \phi(t) d^n t,$$

where

$$A(w, t) = \pi^{-n/4} \prod_{j=1}^{n} \exp\left\{-\frac{1}{2}(w_j^2 + t_j^2) + 2^{1/2} w_j t_j\right\}.$$

On the other hand, in [5] we showed that similar constructions are possible for the Gegenbauer polynomials C_m^λ, $m = 0,1,2,\ldots$, which essentially give the zonal spherical functions on the homogeneous space $SO(n)/SO(n-1) \cong S(\boldsymbol{R}^n)$. That is to say, let F_λ be the Hilbert space of analytic functions f of one complex variable on the unit disk B in $\boldsymbol{C}$, with the inner product given by

$$\langle f, g \rangle_\lambda = \int_B \overline{f(w)} g(w) \rho_\lambda(|w|^2) du dv \ \ (w = u + iv,\ u, v \in \boldsymbol{R}),$$

where

$$\rho_\lambda(t) = \begin{cases} \dfrac{1}{\Gamma(2\lambda - 1)} t^{\lambda-1} \displaystyle\int_t^1 s^{-\lambda}(1-s)^{2\lambda-2} ds & (\lambda > 1/2) \\ t^{\lambda-1}\left\{\dfrac{\Gamma(1-\lambda)}{\Gamma(\lambda)} - \dfrac{1}{\Gamma(2\lambda-1)} \displaystyle\int_0^t s^{-\lambda}(1-s)^{2\lambda-2} ds\right\} & (0 < \lambda \le 1/2), \end{cases}$$

and let K_λ be the usual L^2 space on the open interval $(-1,1)$ with respect to the measure $(1-x^2)^{\lambda-1/2} dx$. Then we have the following proposition (cf.

[5]).

Proposition 1. *A unitary operator,* $f = A_\lambda \phi$, *of* K_λ *onto* F_λ *is defined by*

$$f(w) = \int_{-1}^{1} A_\lambda(w, t)\phi(t)(1-t^2)^{\lambda-1/2}dt,$$

where

$$A_\lambda(w, t) = \frac{2^{\lambda-1/2}\Gamma(\lambda+1)}{\pi} \frac{1-w^2}{(1-2wt+w^2)^{\lambda+1}}$$
$$= \frac{2^{\lambda-1/2}\Gamma(\lambda)}{\pi} \sum_{m=0}^{\infty} (m+\lambda) C_m^\lambda(t) w^m.$$

We should remark that $A_\lambda(w, t)$ can be regarded as a generating function of the Gegenbauer polynomials and the following generating function expansion plays an important role in the proof of this proposition.

$$(1-2wt+w^2)^{-\lambda} = \sum_{m=0}^{\infty} C_m^\lambda(t) w^m, \quad (-1 < t < 1, |w| < 1).$$

As stated above, the Gegenbauer polynomials give the spherical functions on the space $SO(n)/SO(n-1) \cong S(\boldsymbol{R}^n)$, more precisely, for a zonal spherical function ϕ on $SO(n)/SO(n-1) \cong S(\boldsymbol{R}^n)$, there exists a unique nonnegative integer p such that

$$\phi(b) = C_p^{(n-2)/2}(b_1)/C_p^{(n-2)/2}(1),\ b = {}^t(b_1, \dots, b_n) \in S(\boldsymbol{R}^n).$$

Here the identification $SO(n)/SO(n-1) \cong S(\boldsymbol{R}^n)$ is given by $kSO(n-1) \mapsto ke_1$, $k \in SO(n)$ and $e_1 = {}^t(1,0,\dots,0) \in S(\boldsymbol{R}^n)$.

Let us turn to the analogous geometrical object $U(n)/U(n-1) \cong S(\boldsymbol{C}^n)$. Let $H_{p,q}^{(n)}$ be the space of restrictions to $S(\boldsymbol{C}^n)$ of harmonic polynomials $f(\xi, \bar{\xi})$ on $\boldsymbol{C}^n$ which are homogeneous of degree p in ξ and degree q in $\bar{\xi}$. Then it is known (cf. [4], [3]) that $H_{p,q}^{(n)}$ is $U(n)$-invariant and irreducible, and moreover $L^2(U(n)/U(n-1)) = \bigoplus_{p,q=0}^{\infty} H_{p,q}^{(n)}$. In what follows, we denote by $\phi_{p,q}^{(n)}$ the zonal spherical function which belongs to $H_{p,q}^{(n)}$ (cf. [4]).

The purpose of the present paper is to give a construction similar to that for the Hermite or Gegenbauer case for the functions $\phi_{pq}^{(n)}$. The proof will be published elsewhere.

2. Result. Suppose that $n \geq 3$ throughout this section.

Let $\lambda > -1$ and we denote by ρ_λ the function on the open set $(0,1) \times (0,1)$ in $\boldsymbol{R}^2$ defined by

$$\rho_\lambda(u, v) = (uv)^{\lambda/2} \int_1^{\min(1/u, 1/v)} \frac{f_\lambda(tu, tv)}{t} dt,$$

where

$$f_\lambda(u, v) = (uv)^{-\lambda/2}\{(1-u)(1-v)\}^\lambda.$$

Let $\boldsymbol{F}_\lambda$ be the Hilbert space of analytic functions $f(\xi, \eta)$ of two complex variables $(\xi, \eta) \in B \times B$, the direct product of the unit disk B in $\boldsymbol{C}$ with itself, with the inner product defined by

$$\langle f, g \rangle_\lambda = \int_{|\xi|<1} \int_{|\eta|<1} \overline{f(\xi, \eta)} g(\xi, \eta) \rho_\lambda(|\xi|^2, |\eta|^2) d\xi d\eta,$$

where

$$d\xi = d\xi_1 d\xi_2,\ d\eta = d\eta_1 d\eta_2,\ \xi = \xi_1 + i\xi_2,\ \eta = \eta_1 + i\eta_2,\ \xi_j, \eta_j \in \boldsymbol{R},$$

and let $\boldsymbol{K}_\lambda$ be the usual L^2 space on the unit disk B in C with respect to the measure $(1-|z|^2)^{\lambda+1}dxdy$, $z=x+iy$, $x, y \in \boldsymbol{R}$. If we put $\lambda = n-3$, then we have the following:

Theorem 1. *A unitary operator, $f = A_n\varphi$, of $\boldsymbol{K}_{n-3}$ onto $\boldsymbol{F}_{n-3}$ is defined by*

$$f(\xi, \eta) = \int_{|z|<1} A_n(\xi, \eta; z)\varphi(z)(1-|z|^2)^{n-2}dxdy,$$

where

$$A_n(\xi, \eta; z) = \frac{(n-2)(n-1)}{\pi^{3/2}} \frac{1-\xi\eta}{(1-\xi z-\eta\bar{z}+\xi\eta)^n} = \frac{n-2}{\pi^{3/2}} \sum_{p,q=0}^{\infty} (p+q+n-1)R_{pq}^{(n)}(z)\xi^p\eta^q.$$

(*The definitions of the functions $R_{pq}^{(n)}$ will be given in Proposition* 2.)

We only remark that the following proposition in [6], which gives a generating functions $\phi_{pq}^{(n)}$, is a key to solving the problem.

Proposition 2. *If $w, z \in \boldsymbol{C}$, $|w|<1$, $|z| \leq 1$, then*

$$(1-2\mathrm{Re}(wz)+|w|^2)^{1-n} = \sum_{p,q=0}^{\infty} R_{pq}^{(n)}(z)w^p\bar{w}^q,$$

where

$$R_{pq}^{(n)}(b_1) = \binom{n+p-2}{p}\binom{n+q-2}{q}\phi_{pq}^{(n)}(b),\ b = {}^t(b_1, \ldots, b_n) \in S(\boldsymbol{C}^n),$$

and the identification $U(n)/U(n-1) \cong S(\boldsymbol{C}^n)$ is given by $kU(n-1) \mapsto ke_1$, $k \in U(n)$ and $e_1 = {}^t(1,0,\ldots,0) \in S(\boldsymbol{C}^n)$. The series on the right hand side converges absolutely and uniformly for $|z| \leq 1$ and $|w| \leq \rho$ for each $0<\rho<1$.

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