45. *A Space of Siegel Modular Forms Closed under the Action of Hecke Operators*

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In this note, we show that a space of Siegel modular forms whose Fourier coefficients are genus-invariant, is closed under the action of Hecke operators.

Let n be a natural number. We denote the ring of integers by $\boldsymbol{Z}$, the identity matrix of size n by 1_n and the ring of integral square matrices of size n by $M_n(\boldsymbol{Z})$. For matrices A, B, $A[B]$ denotes ${}^tBAB$ if it is well defined. The Siegel upper half space H_n denotes the set of symmetric complex matrices of degree n with positive definite imaginary part. $e(x)$ means $exp(2\pi ix)$ and $\sigma(T)$ denotes the trace of a matrix T.

The definitions of Siegel modular forms, Hecke rings and their action to modular forms are the ordinary ones (see §3.2 in [1]). By using the notation there, our aim is to show the following

Theorem. *Let n, k, q be positive integers and denote by $\mathfrak{M}_k^n(q, \chi)$ the space of Siegel modular forms of degree n, weight k, level q, and Dirichlet character χ modulo q. Put $G_k^n(q, \chi) := \{F(z) = \sum a(T)e(\sigma(Tz)) \in \mathfrak{M}_k^n(q, \chi) \mid a(T)$ depends only on the genus of T if T is positive definite$\}$. Then $G_k^n(q, \chi)$ is closed under the action of the Hecke ring $\mathbf{L}_p^n$ for any prime number p relatively prime to q.*

Remark. The space $G_k^n(q, \chi)$ may be a good one in the sense that it is closed under the Hecke ring. We can give Eisenstein series as examples of Siegel modular forms whose Fourier coefficients are genus-invariant. Another non-trivial example is the Maass space M_k of degree 2 and weight k. If the spaces M_k and $G_k^2(1, 1)$ coincide (this is true when $k = 10$, for example), then it gives a new characterization of the Maass space and it is surprising that the property of being genus-invariant yields the much stronger property. If they are not the same, then it may be worth studying modular forms in $G_k^2(1, 1) \setminus M_k$ in detail.

The theorem is an immediate corollary of the proposition which is given later, by using the result in §3.2 in [1]. Let us give the notion and definition.

Put

$$Sp(n, \boldsymbol{Z}) := \{M \in M_{2n}(\boldsymbol{Z}) \mid {}^tMJ_nM = J_n\}$$

where $J_n := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ and

$$\Gamma_0 := \left\{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in Sp(n, \boldsymbol{Z}) \,\middle|\, \det A = 1\right\}.$$

Let $F(z) := \sum a(T)e(\sigma(Tz))$ be a function on H_n where T runs over the set of rational symmetric matrices of size n, and suppose that it satisfies the the following conditions:

(1) if $a(T) \neq 0$, then T is half-integral and positive semi-definite,

(2) $$F((Az+B)D^{-1}) = F(z) \text{ for every } \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_0.$$

Clearly we have

(3) $$a(T[U]) = a(T) \text{ for } U \in SL_n(\boldsymbol{Z}).$$

We take an integral matrix $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ which satisfies
$$ {}^tMJ_nM = p^\delta J_n$$
where p is a prime number and δ is a natural number. We will fix them hereafter. Let

(4) $$\Gamma_0 M \Gamma_0 = \bigsqcup_i \Gamma_0 \begin{pmatrix} A_i & B_i \\ 0 & D_i \end{pmatrix}$$

be a disjoint coset decomposition, and put
$$(F|'\Gamma_0 M\Gamma_0)(z) := \sum_i F((A_iz+B_i)D_i^{-1}).$$

Proposition. *Suppose that a function $F(z) := \sum a(T)e(\sigma(Tz))$ on H_n satisfy the conditions* (1), (2). *If the value $a(T)$ depends only on the genus of T for every positive definite matrix T, then the same property holds for the Fourier coefficents $a_M(T)$ of $(F|'\Gamma_0M\Gamma_0)(z)$.*

Proof. Let us prove the proposition in the rest.

Lemma 1. *Putting*
$$(F|'\Gamma_0M\Gamma_0)(z) := \sum_T a_M(T)e(\sigma(Tz))$$
and
$$\begin{pmatrix} A_i & B_i \\ 0 & D_i \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} U_i & U_iS_i \\ 0 & {}^tU_i^{-1} \end{pmatrix}$$
for $U_i \in SL_n(\boldsymbol{Z})$, $S_i = {}^tS_i \in M_n(\boldsymbol{Z})$, we have

(5) $$a_M(T) = \sum_i a(p^\delta T[(AU_i)^{-1}])e(\sigma(TS_i))e(\sigma(T[U_i^{-1}]A^{-1}B)).$$

Proof. First we note ${}^tAD = {}^tA_iD_i = p^\delta 1_n$. It is easy to see, $(F|'\Gamma_0M\Gamma_0)(z)$ is equal to
$$\sum_i \sum_T a(T)e(\sigma(T(A_iz+B_i)D_i^{-1}))$$
$$= \sum_{i,T} a(T)e(\sigma(TB_iD_i^{-1}))e(\sigma(D_i^{-1}TA_iz))$$
here by putting $\tilde{T} := D_i^{-1}TA_i = p^{-\delta}\,{}^tA_iTA_i$,
$$= \sum_{i,\tilde{T}} a(D_i\tilde{T}A_i^{-1})e(\sigma(D_i\tilde{T}A_i^{-1}B_iD_i^{-1}))e(\sigma(\tilde{T}z))$$
$$= \sum_{i,\tilde{T}} a(D_i\tilde{T}A_i^{-1})e(\sigma(\tilde{T}A_i^{-1}B_i))e(\sigma(\tilde{T}z)).$$
Hence we have
$$a_M(T) = \sum_i a(D_iTA_i^{-1})e(\sigma(TA_i^{-1}B_i))$$
$$= \sum_i a(D\,{}^tU_i^{-1}T(AU_i)^{-1})e(\sigma(T(AU_i)^{-1}(AU_iS_i + B\,{}^tU_i^{-1})))$$
$$= \sum_i a(DT[U_i^{-1}]A^{-1})e(\sigma(TS_i))e(\sigma(T[U_i^{-1}]A^{-1}B))$$
$$= \sum_i a(p^\delta T[(AU_i)^{-1}])e(\sigma(TS_i))e(\sigma(T[U_i^{-1}]A^{-1}B)).$$

By the condition (1) and Lemma 1, $a_M(T) \neq 0$ implies that $\bar{T} := p^{\delta} T[(AU_i)^{-1}]$ is positive semi-definite and half-integral for some index i. Hence $T = p^{-\delta}\bar{T}[AU_i]$ is positive semi-definite and $2p^{\delta}T$ is an integral matrix. Therefore to prove the proposition, we can confine ourselves to the case that T is a positive definite rational matrix such that

$$2p^{\delta}T \text{ is integral and positive definite.}$$

We take a positive definite matrix T_1 in the genus of T, that is for every prime number q there is a matrix $V_q \in SL_n(\boldsymbol{Z}_q)$ so that

$$T_1 = T[V_q].$$

To prove the proposition and hence the theorem, we have only to show $a_M(T) = a_M(T_1)$. We note that $2p^{\delta}T_1$ is also integral and $\det T_1 = \det T$. We can choose a matrix $V \in SL_n(\boldsymbol{Z})$ so that

$$V \equiv V_q \bmod (2p)^r \boldsymbol{Z}_q \text{ for } q = 2 \text{ and } p, \tag{6}$$

where r is a sufficiently large integer.

Lemma 2. *Putting $T_2 := T[V]$, we have for every i*

$$e(\sigma(T_1[(U_iV)^{-1}]A^{-1}B)) = e(\sigma(T_2[(U_iV)^{-1}]A^{-1}B)), \tag{7}$$

$$e(\sigma(T_1S_i[{}^tV^{-1}])) = e(\sigma(T_2S_i[{}^tV^{-1}])). \tag{8}$$

Moreover, for $T_j' := T_j[(AU_iV)^{-1}]$ $(j = 1,2)$,

$$T_1' \text{ and } T_2' \text{ are in the same genus for any } i. \tag{9}$$

Proof. Because of the condition (6), we have $V_q^{-1}V \equiv 1_n \bmod (2p)^r\boldsymbol{Z}_q$ for $q = 2$ and p. Then $T_2 = T[V] = T_1[V_q^{-1}V]$ implies $2p^{\delta}T_1 \equiv 2p^{\delta}T_2 \bmod (2p)^r\boldsymbol{Z}_q$ because of the integrality of $2p^{\delta}T$, and hence

$$T_1 \equiv T_2 \bmod (2p)^{r-\delta}\boldsymbol{Z}. \tag{10}$$

On the other hand, ${}^tAD = p^{\delta}1_n$ yields that $p^{\delta}A^{-1}$ is an integral matrix. Therefore $(T_1 - T_2)[(U_iV)^{-1}](p^{\delta}A^{-1})B \equiv 0 \bmod (2p)^{r-\delta}\boldsymbol{Z}$ follows, and if $r \geq 2\delta$, then the assertion (7) holds.

The condition (10) also implies (8).

Finally let us prove the assertion (9). Let q be a prime different from $2, p$. Since we have $T_2' = T_2[(AU_iV)^{-1}] = T_1[V_q^{-1}V][(AU_iV)^{-1}] = T_1[V_q^{-1}(AU_i)^{-1}] = T_1'[AU_iVV_q^{-1}(AU_i)^{-1}]$, the fact that A is in $GL_n(\boldsymbol{Z}_q)$ for a prime $q \neq 2, p$ implies that $T_2' = T_1'[W_q]$ for some $W_q \in SL_n(\boldsymbol{Z}_q)$.

Suppose $q = 2$ or p. By virtue of (10), the integrality of $p^{\delta}A^{-1}$ implies $T_1[p^{\delta}(AU_iV)^{-1}] \equiv T_2[p^{\delta}(AU_iV)^{-1}] \bmod (2p)^{r-\delta}$ and hence $T_1' \equiv T_2' \bmod (2p)^{r-2\delta}$. Since $(\det(2p^{\delta}T))(T_j')^{-1} = 2p^{\delta}\det(2p^{\delta}T_j)(2p^{\delta}T_j)^{-1}[{}^t(AU_iV)]$ is integral, we can conclude, using Corollary 5.4.4 in [2] that there is a matrix $W_q \in GL_n(\boldsymbol{Z}_q)$ such that $T_2' = T_1'[W_q]$ if r is sufficiently large. Thus we have shown that there is a matrix $W_q \in GL_n(\boldsymbol{Z}_q)$ such that $T_2' = T_1'[W_q]$ for any prime q. This implies that T_1' and T_2' are in the same genus.

Since

$$\begin{aligned} \Gamma_0 M \Gamma_0 &= \bigsqcup \Gamma_0 M \begin{pmatrix} U_i & U_iS_i \\ 0 & {}^tU_i^{-1} \end{pmatrix} \\ &= \bigsqcup \Gamma_0 M \begin{pmatrix} U_i & U_iS_i \\ 0 & {}^tU_i^{-1} \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & {}^tV^{-1} \end{pmatrix} \end{aligned}$$

$$= \bigsqcup \Gamma_0 M \begin{pmatrix} U_i V & U_i V S_i[{}^t V^{-1}] \\ 0 & {}^t(U_i V)^{-1} \end{pmatrix},$$

(5) implies

$$a_M(T) = a_M(T_2) = \sum_i a(p^\delta T_2[(AU_iV)^{-1}])e(\sigma(T_2 S_i[{}^tV^{-1}]))$$
$$e(\sigma(T_2[(U_iV)^{-1}]A^{-1}B))$$
$$= \sum_i a(p^\delta T_2[(AU_iV)^{-1}])e(\sigma(T_1 S_i[{}^tV^{-1}]))e(\sigma(T_1[(U_iV)^{-1}]A^{-1}B)),$$

using (7) and (8). By the assumption that Fourier coefficients are genus-invariant, the assertion (9) implies $a(p^\delta T_1[(AU_iV)^{-1}]) = a(p^\delta T_2[(AU_iV)^{-1}])$ and hence $a_M(T) = a_M(T_2) = a_M(T_1)$. Thus we have completed the proof of the proposition and hence the theorem.

References

[1] A. N. Andrianov: The multiplicative arithmetic of Siegel modular forms. Russian Math. Survays, **34**, 75–148 (1979).
[2] Y. Kitaoka: Arithmetic of Quadratic Forms. Cambridge University Press (1993).