37. *An Extension of the Derivative of Meromorphic Functions to Holomorphic Curves*

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1. Introduction. A holomorphic curve from $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$ has no notion which plays exactly the same role as the derivative of meromorphic functions. Our purpose of this paper is then to introduce a sort of derivative to holomorphic curves which possesses similar properties to the derivative of meromorphic functions.

Let $f : \boldsymbol{C} \to P^n(\boldsymbol{C})$ be a holomorphic curve and let

$$(f_1, \dots, f_{n+1}) : \boldsymbol{C} \to \boldsymbol{C}^{n+1} - \{0\}$$

be a reduced representation of f, where n is a positive integer. Then, $f_1, \dots, f_{n+1}$ are entire functions without common zeros for all. The characteristic function $T(r, f)$ of f is defined as follows:

$$(1)\qquad T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log \| f(re^{i\theta}) \| \, d\theta - \log \| f(0) \|,$$

where $\| f(z) \| = (| f_1(z) |^2 + \cdots + | f_{n+1}(z) |^2)^{1/2}$. In addition, put

$$(2)\qquad U(z) = \max_{1 \le j \le n+1} | f_j(z) |,$$

then we have the relation

$$(3)\qquad T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log U(re^{i\theta}) d\theta + O(1) \qquad ([1]).$$

It is said that f is transcendental if $\lim_{r\to\infty} T(r, f)/\log r = \infty$. We denote by $\rho(f)$ the order of f:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

and by $S(r, f)$ any quantity satisfying

$$S(r, f) = \begin{cases} O(\log r) \ \ (r \to \infty) & \text{if } \rho(f) < \infty, \\ O(\log rT(r, f)) \ (r \to \infty, \ r \notin E) & \text{if } \rho(f) = \infty, \end{cases}$$

where E is a subset of $[0, \infty)$ for which $m(E) < \infty$.

From now on throughout the paper we suppose that f is non-degenerate; that is to say, the functions $f_1, \dots, f_{n+1}$ are linearly independent over $\boldsymbol{C}$.

Let $W(f_1, \dots, f_{n+1})$ be the Wronskian of $f_1, \dots, f_{n+1}$. Then, it is well-known that $f_1, \dots, f_{n+1}$ are linearly independent over $\boldsymbol{C}$ if and only if $W(f_1, \dots, f_{n+1})$ is not identically equal to zero.

Our definition of an extension of the derivative of meromorphic functions to non-degenerate holomorphic curves is as follows.

Definition (extension of the derivative). we call the holomorphic curve induced by the mapping

$$(4)\qquad (f_1^{n+1},\dots,f_n^{n+1},\ W(f_1,\dots,f_{n+1})):\boldsymbol{C}\to\boldsymbol{C}^{n+1}$$

the derived holomorphic curve of f and express it by f^*.

Note that when $n=1$, f^* corresponds exactly to the derivative of meromorphic function f_2/f_1.

The holomorphic curve f^* has the following properties.

(i) f^* is transcendental if f is transcendental (Theorem 1).

(ii) The order of f^* is equal to the order of f (Theorem 2).

(iii) f^* is not always non-degenerate (Theorem 3).

Applications (see [7]) will appear elsewhere.

We use the standard notation of the Nevanlinna theory of meromorphic functions ([2]).

2. Lemmas. We use the same notation as in Section 1.

Let $d(z)$ be an entire function such that the functions

$$f_1^{n+1}/d,\dots,f_n^{n+1}/d \text{ and } W(f_1,\dots,f_{n+1})/d$$

are entire functions without common zeros for all. Then,

$$(f_1^{n+1}/d,\dots,f_n^{n+1}/d,\ W(f_1,\dots,f_{n+1})/d)$$

is a reduced representation of f^*.

Lemma 1. (a) $T(r,f_k/f_j)<T(r,f)+O(1)\ (k\neq j)$ ([1]).

(b) $T(r,f)<\sum_{j=2}^{n+1}T(r,f_j/f_1)+O(1)$ ([6], Lemme 1).

Remark 1. This lemma holds good even if f is degenerate (see [1] and [6]).

Lemma 2. (a) $W(kf_1,\dots,kf_{n+1})=k^{n+1}W(f_1,\dots,f_{n+1})$;

(b) $\dfrac{W(f_1,\dots,f_{n+1})}{f_1^{n+1}}=W((f_2/f_1)',\dots,(f_{n+1}/f_1)')$,

where $k=k(z)$ is a meromorphic function in $|z|<\infty$ (see [5], p. 108).

Lemma 3. $T(r,f^*)<(n+1)T(r,f)-N(r,1/d)+S(r,f)$.

Proof. Put

$$V(z)=\max\left\{\frac{|f_1(z)|^{n+1}}{|d(z)|},\dots,\frac{|f_n(z)|^{n+1}}{|d(z)|},\frac{|W|}{|d(z)|}\right\},$$

where $W=W(f_1,\dots,f_{n+1})$. Then by (3) we have

$$(4)\qquad T(r,f^*)=\frac{1}{2\pi}\int_0^{2\pi}\log V(re^{i\theta})d\theta+O(1).$$

Put $h_j=f_j/f_1\,(j=2,\dots,n+1)$. Then, since

$$W=f_1\cdots f_{n+1}\cdot\frac{W}{f_1\cdots f_{n+1}}=f_1\cdots f_{n+1}\frac{W(h_2',\dots,h_{n+1}')}{h_2\cdots h_{n+1}}$$

by Lemma 2, we have at any point z

$$|W|\le U(z)^{n+1}\left|\frac{W(h_2',\dots,h_{n+1}')}{h_2\cdots h_{n+1}}\right|$$

and so we have

$$\log V(z)\le(n+1)\log U(z)+\log^+\left|\frac{W(h_2',\dots,h_{n+1}')}{h_2\cdots h_{n+1}}\right|-\log|d(z)|.$$

We then obtain from (4)

$$T(r,f^*)\le\frac{n+1}{2\pi}\int_0^{2\pi}\log U(re^{i\theta})d\theta+\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{W(h_2',\dots,h_{n+1}')}{h_2\cdots h_{n+1}}\right|d\theta$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\log|d(re^{i\theta})|\,d\theta+O(1)$$

$$=(n+1)T(r,f)-N(r,1/d)+S(r,f),$$

where

$$S(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{W(h_2',\ldots,h_{n+1}')}{h_2\cdots h_{n+1}}\right|d\theta+O(1)$$

$$\leq\sum_{k=1}^{n-1}\sum_{j=2}^{n+1}m(r,h_j^{(k)}/h_j)+O(1)$$

$$=\begin{cases}O(\log r)\ \ (r\to\infty) & \text{if } \rho(f)<\infty,\\ O(\log rT(r,f))\ (r\to\infty,\ r\notin E) & \text{otherwise,}\end{cases}$$

E being a subset of $[0,\infty)$ for which $m(E)<\infty$ since by Lemma 1, (a)

(5) $$T(r,h_j)<T(r,f)+O(1)\ (j=2,\ldots,n+1).$$

Lemma 4. *Let $h_1,\ldots,h_n$ be meromorphic in $|z|<\infty$ and linearly independent over $\boldsymbol{C}$.*

(a) *If $\rho(h_j)<\rho(h_n)$ $(j=1,\ldots,n-1)$, then the order of $W(h_1,\ldots,h_n)$ is equal to $\rho(h_n)$* ([4], Lemma 3).

(b) *If $h_1,\ldots,h_{n-1}$ are rational and if h_n is transcendental, then $W(h_1,\ldots,h_n)$ is transcendental.*

The proof of (a) is given in [3], pp. 666-667. Applying the method used in (a) to the case of (b), we can easily prove (b) of this lemma.

3. Results. We use the same notation as in §1 or §2.

Proposition 1. *The definition of f^* is independent of the choice of a reduced representation of f.*

Proof. Let $(g_1,\ldots,g_{n+1})$ be another reduced representation of f. Then there is an entire function $k(z)$ without zero such that $g_j=kf_j$ $(j=1,\ldots,n+1)$. Then, $g_j^{n+1}=k^{n+1}f_j^{n+1}$ $(j=1,\ldots,n)$ and by Lemma 2, (a)

$$W(g_1,\ldots,g_{n+1})=W(kf_1,\ldots,kf_{n+1})=k^{n+1}W(f_1,\ldots,f_{n+1}).$$

Thus we have our proposition.

Proposition 2. $m(r,e_{n+1},f)<T(r,f^*)+N(r,1/d)+S(r,f)$, *where*

$$m(r,e_{n+1},f):=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|f(re^{i\theta})\|}{|f_{n+1}(re^{i\theta})|}d\theta=T(r,f)-N(r,1/f_{n+1}).$$

Proof. Put $W=W(f_1,\ldots,f_{n+1})$ and

$V(z)=\max\{|f_1(z)|^{n+1}/|d(z)|,\ldots,|f_n(z)|^{n+1}/|d(z)|,|W(z)|/|d(z)|\}$,

then it holds that

(6) $$\log V(z)\geq(n+1)\log U(z)-\sum_{j=1}^{n}\log^+\left|\frac{f_{n+1}(z)}{f_j(z)}\right|-\log^+\left|\frac{f_1(z)\cdots f_{n+1}(z)}{W(z)}\right|-\log|d(z)|,$$

where $U(z)$ is defined in (2). In fact,

(i) When $U(z)=\max\{|f_1(z)|,\ldots,|f_n(z)|\}$, it is easy to see that

(7) $$\log V(z)\geq(n+1)\log U(z)-\log|d(z)|.$$

(ii) When $U(z)=|f_{n+1}(z)|$,

(8) $$\log|W(z)|=\log\left|\frac{W(z)}{f_1(z)\cdots f_{n+1}(z)}\right|$$

$$+ \log \left| \frac{f_1(z) \cdots f_{n+1}(z)}{f_{n+1}(z)^{n+1}} \right| + (n+1) \log |f_{n+1}(z)|$$
$$= (n+1) \log U(z) - \sum_{j=1}^{n} \log^+ \left| \frac{f_{n+1}(z)}{f_j(z)} \right|$$
$$+ \log^+ \left| \frac{W(z)}{f_1(z) \cdots f_{n+1}(z)} \right| - \log^+ \left| \frac{f_1(z) \cdots f_{n+1}(z)}{W(z)} \right|$$
$$\geq (n+1) \log U(z) - \sum_{j=1}^{n} \log^+ \left| \frac{f_{n+1}(z)}{f_j(z)} \right| - \log^+ \left| \frac{f_1(z) \cdots f_{n+1}(z)}{W(z)} \right|.$$

Since

$$\log V(z) \geq \log |W(z)| - \log |d(z)|,$$

we have (6) from (7) and (8). We have from (3) and (6)

$$T(r, f^*) + O(1) \geq (n+1) T(r, f) - \sum_{j=1}^{n} m\left(r, \frac{f_{n+1}}{f_j}\right) - m\left(r, \frac{f_1 \cdots f_{n+1}}{W}\right) - N(r, 1/d)$$

by the first fundamental theorem of Nevanlinna,

$$\geq (n+1) T(r, f) - \sum_{j=1}^{n} m\left(r, \frac{f_{n+1}}{f_j}\right) - m\left(r, \frac{W}{f_1 \cdots f_{n+1}}\right) - N\left(r, \frac{W}{f_1, \ldots, f_{n+1}}\right) - N(r, 1/d) + O(1)$$
$$\geq (n+1) T(r, f) - \sum_{j=1}^{n} m\left(r, \frac{f_{n+1}}{f_j}\right) - \sum_{j=1}^{n+1} N\left(r, \frac{1}{f_j}\right) - N(r, 1/d) - S(r, f)$$
$$\geq T(r, f) - N(r, 1/f_{n+1}) - N(r, 1/d) - S(r, f)$$
$$= m(r, \boldsymbol{e}_{n+1}, f) - N(r, 1/d) - S(r, f)$$

since

$$T(r, f) + O(1) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\| f \|}{|f_j|} d\theta + N\left(r, \frac{1}{f_j}\right)$$
$$\geqq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{|f_{n+1}|}{|f_j|} d\theta + N\left(r, \frac{1}{f_j}\right)$$
$$= m(r, f_{n+1}/f_j) + N(r, 1/f_j) \quad (j = 1, \ldots, n).$$

Theorem 1. f^* *is transcendental if f is transcendental.*

Proof. (a) If there is at least one f_j $(2 \leq j \leq n)$ such that f_j/f_1 is transcendental, then $f_j^{n+1}/f_1^{n+1} = (f_j/f_1)^{n+1}$ is transcendental, so that by Lemma 1, (a), f^* is transcendental.

(b) Suppose that f_j/f_1 $(j = 2, \ldots, n)$ are rational. Then, since f is transcendental, f_{n+1}/f_1 is transcendental by Lemma 1, (b). We want to prove that W/f_1^{n+1} is transcendental.

Now, by Lemma 2, (b)

(9) $$W/f_1^{n+1} = W((f_2/f_1)', \ldots, (f_{n+1}/f_1)').$$

Applying Lemma 4, (b) to $h_j = f_{j+1}/f_1 (j = 1, \ldots, n)$ in (9), we obtain that W/f_1^{n+1} is transcendental. This shows that f^* is transcendental.

Theorem 2. $\rho(f^*) = \rho(f)$.

Proof. By Lemma 3, it is easy to see that $\rho(f^*) \leq \rho(f)$. We note that $\rho(f_j/f_1) \leq \rho(f)$ $(j = 2, \ldots, n+1)$ by Lemma 1, (a).

(a) Suppose that there is at least one f_j $(2 \leq j \leq n)$ such that $\rho(f_j/f_1) =$

$\rho(f)$. Then, $\rho(f_j^{n+1}/f_1^{n+1}) = \rho((f_j/f_1)^{n+1}) = \rho(f)$. By Lemma 1, (a), we have $\rho(f^*) \geq \rho(f)$.

(b) Suppose that $\rho(f_j/f_1) < \rho(f)$ $(j = 2, \ldots, n)$. Then, by Lemma 1, (b), $\rho(f_{n+1}/f_1) = \rho(f)$. We want to prove that $\rho(W/f_1^{n+1}) = \rho(f)$.

Now, by Lemma 2, (b), (9) holds. Applying Lemma 4, (a) to $h_j = f_{j+1}/f_1$ $(j = 1, \ldots, n)$ in (9), we obtain that $\rho(W/f_1^{n+1}) = \rho(f)$, since $\rho(h_j) < \rho(f)$ $(j = 1, \ldots, n-1)$ and $\rho(h_n) = \rho(f)$. This and Lemma 1 (a) show that $\rho(f^*) \geq \rho(f)$.

Thus we have $\rho(f^*) = \rho(f)$.

Theorem 3. *f^* is not always non-degenerate.*

Proof. We have only to give an example of non-degenerate, transcendental holomorphic curve whose derived holomorphic curve is degenerate.

Let m be any integer not smaller than 3 and put $n = 2m - 1$. Put

$$f_j = e^{(2j-1)z/2m}\ (j = 1, \ldots, m),\ f_{m+1} = e^z + 1,\ f_{m+2} = e^z - 1 \text{ and}$$
$$f_{m+j} = z^{j-2}\quad (j = 3, \ldots, m).$$

Then, it is easy to see that these $n + 1 = 2m$ entire functions have no common zeros and are linearly independent over C. Let f be the holomorphic curve induced by the mapping

$$(f_1, \ldots, f_{2m}) : C \to C^{2m}.$$

Then, f is non-degenerate and it is easy to see that f is transcendental by Lemma 1, (a).

Now, f^* is induced by

$$(f_1^{2m}, \ldots, f_{2m-1}^{2m}, W(f_1, \ldots, f_{2m})).$$

In this case, $f_1^{2m}, \ldots, f_{m+2}^{2m}$ are linearly dependent over C.

In fact

$$f_{m+1}^{2m} - f_{m+2}^{2m} = (e^z + 1)^{2m} - (e^z - 1)^{2m} = 2\sum_{j=1}^{m} {}_{2m}C_{2j-1} e^{(2j-1)z}$$
$$= 2\sum_{j=1}^{m} {}_{2m}C_{2j-1} f_j^{2m}.$$

This shows that f^* is degenerate.

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