31. *Crepant Resolution of Trihedral Singularities*

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§1. Introduction. The purpose of this paper is to construct a crepant resolution of quotient singularities by finite subgroups of $SL(3, \mathbf{C})$ of certain type, and prove that each Euler number of the minimal model is equal to the number of conjugacy classes.

The problem of finding a nice resolution of quotient singularities by finite subgroups of $SL(3, \mathbf{C})$ arose from mathematical physics. In the superstring theory, the dimension of the space-time is 10, four of them are usual space and time dimensions, and other six are compactified on a compact Calabi-Yau space M. From a point of view of algebraic geometry, the Calabi-Yau space is a smooth three-dimensional complex projective variety whose canonical bundle is trivial and fundamental group is finite.

In the physics of superstring theory, one considers the string propagation on a manifold M which is a quotient by a finite subgroup of symmetries G. By a physical argument of string vacua of M/G, one concludes that the correct Euler number for the theory should be the "orbifold Euler characteristic" [3], defined by

$$\chi(M, G) = \frac{1}{|G|} \sum_{gh=hg} \chi(M^{\langle g,h \rangle}),$$

where the summation runs over all pairs of commuting elements of G, and $M^{\langle g,h \rangle}$ denotes the common fixed set of g and h. For the physicist's interest, we only consider M whose quotient space M/G has trivial canonical bundle.

Conjecture I ([3]). *There exists a resolution of singularities* $\widetilde{M/G}$ *s.t.* $\omega_{\widetilde{M/G}} \simeq \mathcal{O}_{\widetilde{M/G}}$, *and*

$$\chi(\widetilde{M/G}) = \chi(M, G).$$

This conjecture follows from its local form [6]:

Conjecture II (local form). *Let* $G \subset SL(3, \mathbf{C})$ *be a finite group. Then there exists a resolution of singularities* $\sigma : \tilde{X} \to \mathbf{C}^3/G$ *with* $\omega_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}$ *and*

$$\chi(\tilde{X}) = \#\{\textit{conjugacy class of } G\}.$$

In algebraic geometry, the conjecture says that a minimal model of the quotient space by a finite subgroup of $SL(3, \mathbf{C})$ is non-singular.

Conjecture II was proved for abelian groups by Roan ([18]), and independently by Markushevich, Olshanetsky and Perelomov ([11]) by using toric method. It was also proved for 5 other groups, for which X are hypersurfaces: (i) WA_3^+, WB_3^+, WC_3 where WX^+ denotes the positive determinant part of the Weyl group WX of a root system X by Bertin and Markushevich ([1]), (ii) H_{168} by Markushevich ([10]), and (iii) I_{60} by Roan ([19]).

In this paper, we prove Conjecture II for solvable groups of certain type:

Definition. *Trihedral group* is a finite group $G = \langle H, T \rangle \subset SL(3,\mathbf{C})$, where $H \subset SL(3, \mathbf{C})$ is a finite group generated by diagonal matrices and

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Definition. *Trihedral singularities* are quotient singularities by trihedral groups.

Definition. A resolution of singularities $f: Y \to X$ of a normal variety X is *crepant* if $K_Y = f^*K_X$.

Theorem 1.1 (Main Theorem). *Let $X = \mathbf{C}^3/G$ be a quotient space by a trihedral group G. Then there exists a crepant resolution of singularities*

$$f: \tilde{X} \to X,$$

and

$$\chi(\tilde{X}) = \#\{\textit{conjugacy class of } G\}.$$

Trihedral singularities are 3-dimensional version of D_n-singularities, and they are non-isolated and many of them are not complete intersections. Their resolutions are similar to those of D_n-singularities. There is a nice combination of the toric resolution and Calabi-Yau resolution.

By the way, the conjecture II is true in dimension 2 (i.e., the case of $SL(2, \mathbf{C})$ (cf. [6])), but in the case of $SL(4, \mathbf{C})$ there exists a counter-example; in the case of group $G = \langle [-1, -1, -1, -1] \rangle$ (diagonal matrix), which is a finite subgroup of $SL(4, \mathbf{C})$, but there isn't a crepant resolution.

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§2. Idea for proof. Before the proof of Main Theorem, we recall a minimal resolution of D_n-singularity in dimension 2. It is a quotient singularity by a binary dihedral group G generated by

$$H = \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\delta = \exp(2\pi\sqrt{-1}/(2m))$ and $m = n - 2$.

Let (u, v) be a coordinate of $\mathbf{C}^2$. Then the invariant ring under the action of G is

$$\mathbf{C}[(u^{2m} - v^{2m})uv, u^{2m} + v^{2m}, u^2v^2] \cong \mathbf{C}[x, y, z]/(x^2 - y^2z + 4z^{m+1}).$$

Then we can construct a minimal resolution as follows.

$$\begin{array}{ccccc} & & & & \tilde{X} \\ & & & & \downarrow \tau \\ & & \tilde{Y} & \xrightarrow{/\mathfrak{A}_2} & \tilde{Y}/\mathfrak{A}_2 \\ & & \downarrow \pi & & \downarrow \tilde{\pi} \\ \mathbf{C}^2 & \longrightarrow & \mathbf{C}^2/\mathbf{Z}_{2m} = Y & \xrightarrow{/\mathfrak{A}_2} & \mathbf{C}^2/G = X \end{array}$$

where Y has a A_{2m-1}-singularity.

(1) At first, we construct a minimal resolution of Y whose exceptional divisor as follows.

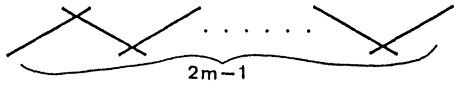

Fig. (2.1)

(2) And the action of $\mathfrak{A}_2$ gives an involution.

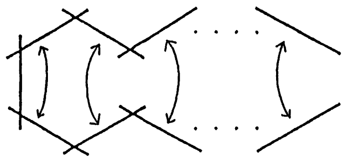

Fig. (2.2)

(3) So we identify the corresponding two curves. Then we have two singularities on the quotient of the central curve by $\mathfrak{A}_2$.

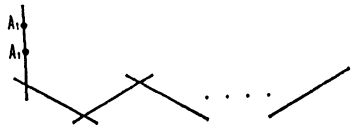

Fig. (2.3)

(4) So, we resolve the singularity, then we obtain a resolution of D_n-singularity.

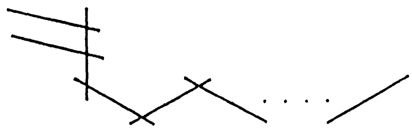

Fig. (2.4)

§3. Crepant resolution of trihedral singularities G'. Let G' be the subgroup of the group $G = \langle H, T\rangle$ consisting of all the diagonal matrices. Then G' is a normal subgroup, and an abelian group. We consider the order of G'.

Proposition 3.1. *$|G'|$ is one of the following holds.*

(1) $|G'| \equiv 0$ (*mod* 3)

(2) $|G'| \equiv 1$ (*mod* 3).

From now, we call the type of G' as the following:

Type (I) when $|G'| \equiv 1$ (mod 3)

Type (II) when $|G'| \equiv 0$ (mod 3).

Proposition 3.2. *Let $X = \mathbf{C}^3/G$, and $Y = \mathbf{C}^3/G'$. Then there exists the* *following diagram:*

$$\begin{array}{ccccc} & & & & \tilde{X} \\ & & & & \downarrow \tau \\ & & \tilde{Y} & \xrightarrow{\tilde{\mu}} & \tilde{Y}/\mathfrak{A}_3 \\ & & \downarrow \pi & & \downarrow \tilde{\pi} \\ \mathbf{C}^3 & \longrightarrow & \mathbf{C}^3/G' = Y & \xrightarrow{\mu} & \mathbf{C}^3/G = X \end{array}$$

where π is a resolution of the singularity of Y, and $\tilde{\pi}$ is the induced morphism, τ is a resolution of the singularity by $\mathfrak{A}_3$, and $\tau \circ \tilde{\pi}$ is a crepant resolution of the singularity of X.

Sketch of the proof. As a resolution π of Y, we take a toric resolution, which is also crepant. Then we lift the $\mathfrak{A}_3$-action on Y to its minimal resolution $\tilde{Y}$ and form the quotient $\tilde{Y}/\mathfrak{A}_3$. This quotient gives in a natural way a partial resolution of the singularities of X. The minimal resolution $\tilde{X} \to \tilde{Y}/\mathfrak{A}_3$ of the singularities of $\tilde{Y}/\mathfrak{A}_3$ induces a complete resolution of X.

Under the action of $\mathfrak{A}_3$, the singularities of $\tilde{Y}/\mathfrak{A}_3$ lie in the union of the image of the exceptional divisor of $\tilde{Y}$ under $\tilde{Y} \to \tilde{Y}/\mathfrak{A}_3$ and the locus C : $(x = y = z)$.

In the resolution $\tilde{Y}$ of Y, the group $\mathfrak{A}_3$ permutes exceptional divisors. So the fixed points on the exceptional divisors consist of one point or three points.

Claim I. *There exists a toric resolution of Y where $\mathfrak{A}_3$ acts symmetrically on the exceptional divisors.*

Claim II. *Let X_S be the corresponding torus embedding, then X_S is non-singular.*

We obtain a crepant resolution $\pi_S : X_S \to \mathbf{C}^3/G'$.

Claim III. *Let F be a fixed locus on $\tilde{Y}$ under the action of $\mathfrak{A}_3$, then*

$$F := \begin{cases} C & \text{if } G' \text{ is type } (I) \\ C \cup \{2 \text{ points}\} & \text{if } G' \text{ is type } (II) \end{cases}$$

where C is a strict transform of the fixed locus in Y.

$\mathfrak{A}_3$-action in the neighbourhood of a fixed point is analytically isomorphic to some linear action.

Claim IV. *Let $Z = \mathbf{C}^3/\mathfrak{A}_3$, then* $\chi(\tilde{Z}) = \chi(\mathbf{C}^3, \mathfrak{A}_3) = 3$.

Claim V. *The resolution $\tau \circ \tilde{\pi}$ is a crepant resolution.*

Lemma 3.3. *Let $X := \mathbf{C}^3/\langle G', T\rangle$, and $f : \tilde{X} \to X$ the crepant resolution as above. Then the Euler number of $\tilde{X}$ is given by*

$$\chi(\tilde{X}) = \frac{1}{3}(|G'| - k) + 3k$$

where

$$k = \begin{cases} 1 & \text{if } |G'| \equiv 1 (\mathrm{mod}\ 3) \ (\text{type } (I)) \\ 3 & \text{if } |G'| \equiv 0 (\mathrm{mod}\ 3) \ (\text{type } (II)) \end{cases}.$$

Theorem 3.4. $\chi(\tilde{X}) = \#\{\text{conjugacy class of } G\}$.

§4. Example. In this section, we will see an example.

Example. Let G be a group generated by $[1, -1, -1]$ and T. Then the normal subgroup will be $G' = \langle [1, -1, -1], [-1, -1, 1] \rangle$, i.e., G' is Type (I).

(1) The dual graph of toric resolution of $Y = \mathbf{C}^3/G'$ is one of the following.

(a) Fig. (4.1) (b)

(a) is $\mathfrak{A}_3$ invariant, while (b) is not. So we take (a).

(2) By the action of $\mathfrak{A}_3$ on $\tilde{Y}$, three of the four triangles are permuted, and there is one triangle corresponding one point which is fixed by $\mathfrak{A}_3$.

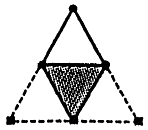

Fig. (4.2)

(3) By the resolution of the singularities in $\tilde{Y}/\mathfrak{A}_3$, the central component is replaced by two $\mathbf{P}^1$-bundle interesting at their sections, whose Euler number is 3.

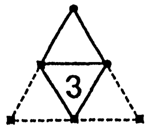

Fig. (4.3)

(4) Eular characteristics of the minimal model.

$$\chi(\tilde{X}) = 1 + 3 = 4.$$

(5) Conjugacy class of G. There are 4 conjugacy classes;

$$e, [T], [T^2], [[1, -1, -1]].$$

Therefore

$$\chi(\mathbf{C}^3, G) = 4.$$

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