24. On Confluences of the General Hypergeometric Systems

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(Communicated by Kiyosi ITÔ, M. J. A., May 12, 1993)

Introduction. Let r and $n(>r)$ be positive integers and let $Z_{r,n}$ be the set of $r \times n$ complex matrices of maximal rank.

In the preceding paper [7], we introduced, for any given composition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of the integer n, the generalized confluent hypergeometric functions with variables $z = (z_{ip})_{0 \le i \le r-1, 0 \le p \le n-1} \in Z_{r,n}$. They are defined as solutions of the system of partial differential equations on $Z_{r,n}$ called the generalized confluent hypergeometric system (see Definition 1.1). In case where the composition of n is $\lambda = (1, \ldots, 1)$, the generalized confluent hypergeometric functions coincide with the general hypergeometric functions due to K. Aomoto and I. M. Gelfand ([1], [2]).

One may ask why we have given the name "the generalized confluent" hypergeometric functions to the functions we introduced. The purpose of this paper is to justify our naming to these functions. In fact we show that the generalized confluent hypergeometric systems can be obtained from the Aomoto-Gelfand's system by a finite number of certain limit processes called the processes of confluence (see Theorem 2.5). It is to be noted that the processes of confluence for our systems are determined from the group theoretic point of view (Theorems 2.3, 2.4).

To explain our problem more concretely, we recall a classical example: a confluence of two singular points for the hypergeometric equation of Gauss. The Gauss hypergeometric equation is

(0.1) $x(1-x)u'' + \{\gamma - (\alpha+\beta+1)x\}u' - \alpha\beta u = 0, \quad ' = d/dx.$

For the equation (0.1), consider a change of variables and parameters

$$x = \varepsilon\xi, \quad \beta = 1/\varepsilon.$$

Then the equation for (ξ, u) is

$$\xi(1-\varepsilon\xi)\frac{d^2u}{d\xi^2} + (\gamma - \varepsilon(\alpha + \varepsilon^{-1} + 1)\xi)\frac{du}{d\xi} - \alpha u = 0 \tag{0.2}$$

and the coefficients of $d^2u/d\xi^2$, $du/d\xi$ and u depend holomorphically on ε at $\varepsilon = 0$. Putting $\varepsilon = 0$ in the equation (0.2) we obtain the Kummer's confluent hypergeometric equation

$$\xi\frac{d^2u}{d\xi^2} + (\gamma - \xi)\frac{du}{d\xi} - \alpha u = 0. \tag{0.3}$$

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Our problem is to generalize this process for all confluent hypergeometric systems.

In Section 1 we recall the definition of the generalized confluent hypergeometric functions (system) and their properties necessary in stating our results of this paper. In Section 2, we give our main results. We also explain the above process from the Gauss hypergeometric equation to the Kummer's confluent hypergeometric one in the framework of our general confluence process.

1. Generalized confluent hypergeometric systems. There is given a composition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n, i. e. the sequence of positive integers satisfying $\lambda_1 + \cdots + \lambda_l = n$ and $\lambda_k (k = 1, \ldots, l)$ are not necessarily arrayed in descending or ascending order. We associate a composition λ of n with a diagram defined as the set of points $(k, m) \in \mathbf{Z}^2$ such that $0 \leq m \leq \lambda_k - 1$. In drawing such diagram we adopt the convention that the first coordinate k (the row index) increases as one goes downwards, and the second coordinate m (the column index) increases as one goes from left to right, and we place a square at each point $(k, m) \in \mathbf{Z}^2$ as is illustrated in the figure. The number of squares of the diagram λ will be called the weight of λ and will be denoted by $|\lambda|$.

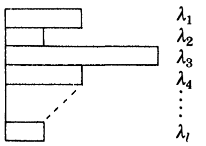

Fig. 1

Definition 1.1. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a diagram of weight n and let $\alpha = {}^t(\alpha^{(1)}, \ldots, \alpha^{(l)})$, $\alpha^{(k)} = (\alpha_0^{(k)}, \alpha_1^{(k)}, \ldots, \alpha_{\lambda_k - 1}^{(k)})$ $(k = 1, \ldots, l)$, be a constant column vector of dimension n satisfying $\sum_{k=1}^{l} \alpha_0^{(k)} = -r$. The generalized confluent hypergeometric system of type λ (CHG system for short) is the system of linear partial differential equations:

$$M_\lambda : \begin{cases} L_{km} u = \alpha_m^{(k)} u, & k = 1, \ldots, l\,;\ m = 0, \ldots, \lambda_k - 1 \\ M_{ij} u = -\delta_{ij} u, & i, j = 0, \ldots, r-1 \\ \square_{ij,pq} u = 0 & i, j = 0, \ldots, r-1\,;\ p, q = 0, \ldots, n-1 \end{cases}$$

where

$$L_{km} = \sum_{q=0}^{r-1} \sum_{\substack{\lambda_1 + \cdots + \lambda_{k-1} + m \\ \leq p \leq \lambda_1 + \cdots + \lambda_k - 1}} z_{q,p-m} \frac{\partial}{\partial z_{qp}},$$

$$M_{ij} = \sum_{p=0}^{n-1} z_{ip} \frac{\partial}{\partial z_{jp}},$$

$$\square_{ij,pq} = \frac{\partial^2}{\partial z_{ip} \partial z_{jq}} - \frac{\partial^2}{\partial z_{iq} \partial z_{jp}},$$

δ_{ij} being the Kronecker's symbel. We denote the system by $M_\lambda(\alpha)$ or $M_\lambda(z\,;\alpha)$, if it is necessary to indicate the constants α or, moreover, the independent variables $z = (z_{ip})$.

It has been shown ([7]) that the system M_λ is holonomic and hence the space of solutions for M_λ at a "generic" point of $Z_{r,n}$ is isomorphic to a finite dimensional complex vector space. Any solution of the system M_λ is called *the generalized confluent hypergeometric function of type* λ (CHG function for short).

Now we recall some properties of the CHG functions of type λ. We first explain several terminologies.

Let $J(m)$ be the Jordan group of size m, namely, a maximal abelian Lie subgroup of $GL(m, \boldsymbol{C})$ defined by

$$J(m) := \left\{c = \sum_{i=0}^{m-1} c_i \Lambda^i \,;\, c_i \in \boldsymbol{C}, \quad c_0 \neq 0\right\},$$

where $\Lambda = \Lambda_m = (\delta_{i+1,j})_{0 \le i,j \le m-1} \in M(m, \boldsymbol{C})$. The group $J(m)$ is isomorphic to the group $\boldsymbol{C}^\times \times \boldsymbol{C}^{m-1}$, where $\boldsymbol{C}^{m-1}$ is equipped with the natural additive structure. In fact, this isomorphism is established by associating $c = \sum_{i=0}^{m-1} c_i \Lambda^i$ with $(c_0, \theta_1(c), \ldots, \theta_{m-1}(c))$.
Here

$$\theta_i(c) := \theta_i((c_0, c_1, \ldots, c_{m-1}, 0, \ldots)), \quad i \ge 0,$$

where $\theta_i (i \ge 0)$ are functions defined by

$$\begin{aligned}\sum_{i=0}^{\infty} \theta_i(x) t^i &= \log(x_0 + x_1 t + x_2 t^2 + \cdots) \\ &= \log x_0 + \log(1 + \frac{x_1}{x_0} t + \frac{x_2}{x_0} t^2 + \cdots)\end{aligned}$$

for a sequence $x = (x_0, x_1, x_2, \ldots)$ with $x_0 \neq 0$. We note that $\theta_0(c) = \log c_0$ and $\theta_i(c)$ $(i \ge 1)$ is a weighted homogeneous polynomial of $c_1/c_0, \ldots, c_i/c_0$. It follows from this fact that a character $\chi : \tilde{J}(m) \to \boldsymbol{C}^\times$ of the universal covering group $\tilde{J}(m)$ of $J(m)$ is written as

$$\chi(c) = \exp\left(\sum_{i=0}^{m-1} \alpha_i \theta_i(c)\right) = c_0^{\alpha_0} \exp\left(\sum_{i=1}^{m-1} \alpha_i \theta_i(c)\right), \quad c = \sum_{i=0}^{m-1} c_i \Lambda^i$$

for some $\alpha = {}^t(\alpha_0, \ldots, \alpha_{m-1}) \in \boldsymbol{C}^m$. This character will be denoted by $\chi(\cdot\,; \alpha)$ to indicate the dependence on α.

For a diagram $\lambda = (\lambda_1, \ldots, \lambda_l)$ of weight n, we consider an abelian Lie subgroup $H_\lambda = J(\lambda_1) \times \cdots \times J(\lambda_l) \subset GL(n, \boldsymbol{C})$ of dimension n and its universal covering group $\tilde{H}_\lambda$. Let $\alpha = {}^t(\alpha^{(1)}, \ldots, \alpha^{(l)}) \in \boldsymbol{C}^n$ be the constant column vector given in Definition 1.1. Then $\chi_\lambda(\cdot\,; \alpha) : \tilde{H}_\lambda \to \boldsymbol{C}^\times$ defined by $\chi_\lambda(c\,; \alpha) = \Pi_k \chi(c^{(k)}\,; \alpha^{(k)})$ for

$$c = (c^{(1)}, \ldots, c^{(l)}) \in \tilde{H}_\lambda, \; c^{(k)} = \sum_{i=0}^{\lambda_k - 1} c_i^{(k)} \Lambda_{\lambda_k}^i \in \tilde{J}(\lambda_k)$$

is a character of $\tilde{H}_\lambda$ called the character of homogeneity $\alpha \in \boldsymbol{C}^n$.

As one of the important properties for the CHG functions, we have

Proposition 1.2. *The generalized confluent hypergeometric function* $\Phi_\lambda(z\,; \alpha)$ *of type* λ *satisfies*

$$\text{(1.1)} \qquad \begin{cases} \phi_\lambda(zc\,; \alpha) = \Phi_\lambda(z\,; \alpha)\chi_\lambda(c\,; \alpha) & \text{for } c \in H_\lambda. \\ \phi_\lambda(gz\,; \alpha) = (\det g)^{-1}\Phi_\lambda(z\,; \alpha) & \text{for } g \in GL(r, \boldsymbol{C}). \end{cases}$$

The first and the second properties in (1.1) are called the H_λ-homogeneity and the $GL(r, \boldsymbol{C})$-homogeneity for $\Phi_\lambda(z\,; \alpha)$, respectively.

The CHG system M_λ admits solutions in an integral representation. Let us define a biholomorphic mapping

$$\iota_\lambda : H_\lambda \to \prod_{k=1}^{l} (C^\times \times C^{\lambda_k - 1}) \subset C^n$$

by

$$\iota_\lambda(c) = (c_0^{(1)}, \ldots, c_{\lambda_1-1}^{(1)}, \ldots, c_0^{(l)}, \ldots, c_{\lambda_l-1}^{(l)}) \quad \text{for}$$

$$c = \bigoplus_{k=1}^{l} \sum_{i=1}^{\lambda_k-1} c_i^{(k)} \Lambda_{\lambda_k}^i \in H_\lambda \subset GL(n, C).$$

For $t = (t_0, \ldots, t_{r-1}) \in C^r$ and for $z = (z_0, \ldots, z_{n-1}) \in Z_{r,n}$, we set

$$\langle t, z \rangle = (\langle t, z_0 \rangle, \ldots, \langle t, z_{n-1} \rangle),$$

where

$$\langle t, z_j \rangle = \sum_{i=0}^{r-1} t_i z_{ij} \ (0 \leq j \leq n-1).$$

Let ω be the $(r-1)$-form defined by

$$\omega := \sum_{i=0}^{r-1} (-1)^i t_i dt_0 \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \cdots \wedge dt_{r-1}.$$

Proposition 1.3 [7]. *For an appropriate $(r-1)$-dimensional cycle Δ in C^r, the integral*

$$\Phi_\lambda(z ; \alpha) = \int_\Delta \chi_\lambda(\iota_\lambda^{-1}(\langle t, z \rangle) ; \alpha)\omega$$

gives a solution of the CHG system M_λ.

There is a trivial symmetries for the CHG systems. To describe it we change the manner of indexing of $z \in Z_{r,n}$ as

$$z = (z^{(1)}, \ldots, z^{(l)}), \quad z^{(k)} = (z_0^{(k)}, z_1^{(k)}, \ldots, z_{\lambda_k-1}^{(k)}) \quad (k = 1, \ldots, l).$$

Let $\mathfrak{S}_l$ be the symmetric group on the set $\{1, \ldots, l\}$ and define the action of $\sigma \in \mathfrak{S}_l$ on (λ, α, z) by

$$\sigma \cdot \lambda = (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(l)}), \ \sigma \cdot \alpha = (\alpha^{(\sigma(1))}, \ldots, \alpha^{(\sigma(l))}), \quad \sigma \cdot z = (z^{(\sigma(1))}, \ldots, z^{(\sigma(l))}).$$

Proposition 1.4. *Let λ be a diagram of weight n. Then the change of variables $(\alpha, z) \mapsto (\alpha', z') = (\sigma \cdot \alpha, \sigma \cdot z)$ takes the CHG system $M_\lambda(\alpha)$ to the system $M_{\sigma \cdot \lambda}(\sigma \cdot \alpha)$.*

This trivial result says that if the diagram μ is obtained from λ by permuting its rows, the systems M_λ and M_μ are essentially the same.

2. Main results.

Definition 2.1. Let λ and μ be diagrams of weight n. The diagram μ is said to be *adjacent* to λ if μ is obtained from λ by making some two rows of λ into a single row whose length is the sum of those of the two rows of λ. We denote this relation by the symbol $\lambda \to \mu$.

Example 2.2. For the diagrams of weight 4, the following figure describes all the relations of adjacency among them mudulo permutations of rows in the diagrams.

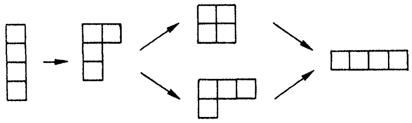

Fig. 2

Let $\mu = (\mu_1, \ldots, \mu_{l-1})$ be adjacent to $\lambda = (\lambda_1, \ldots, \lambda_l)$ of weight n. We may suppose that $\mu_1 = \lambda_1, \ldots, \mu_{h-1} = \lambda_{h-1}, \mu_h = \lambda_h + \lambda_{h+1}, \mu_{h+1} = \lambda_{h+2}, \ldots, \mu_{l-1} = \lambda_l$ for some $1 \leq h < l$. We define a holomorphic mapping $g = g_{\lambda \to \mu} : C \setminus \{0\} \to$

$GL(n, \boldsymbol{C})$ by $g(\varepsilon) = (g_{ij}(\varepsilon))_{1\leq i,j\leq l}$, where $g_{ij}(\varepsilon)$ is a $\lambda_i \times \lambda_j$ matrix given by
$g_{ii}(\varepsilon) = I_{\lambda_i}, \quad i \neq h+1$

$$\begin{pmatrix} g_{h+1}(\varepsilon) \\ g_{h+1}(\varepsilon) \end{pmatrix} = D_{\lambda_h+\lambda_{h+1}}(\varepsilon) \begin{pmatrix} \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{\lambda_h-1} \\ \binom{1}{0} & \binom{1}{1} & \cdots & \binom{1}{\lambda_h-1} \\ \vdots & \vdots & \cdots & \vdots \\ \binom{\lambda_h+\lambda_{h+1}-1}{0} & \binom{\lambda_h+\lambda_{h+1}-1}{1} & \cdots & \binom{\lambda_h+\lambda_{h+1}-1}{\lambda_{h+1}-1} \end{pmatrix} D_{\lambda_{h+1}}(\varepsilon^{-1})$$

$g_{ij}(\varepsilon) = 0$, otherwise,
$D_m(\varepsilon)$ denoting diag $(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{m-1})$. Remark that det $g(\varepsilon) = \varepsilon^{\lambda_h\lambda_{h+1}}$.

Now we state our results. Let μ be adjacent to λ. Define $H_{\lambda\to\mu,\varepsilon}$ by

$$H_{\lambda\to\mu,\varepsilon} := \{\iota_\lambda^{-1}(\iota_\mu(c)g(\varepsilon)) \in H_\lambda \,;\, c \in H_\mu\},$$

where $g(\varepsilon) = g_{\lambda\to\mu}(\varepsilon)$. Then we have

Theorem 2.3 (confluence for H_λ). *It holds that*

$$\lim_{\varepsilon\to 0} Ad_{g(\varepsilon)} H_{\lambda\to\mu,\varepsilon} = H_\mu,$$

more precisely, for every $c \in H_\mu$,

$$Ad_{g(\varepsilon)}\iota_\lambda^{-1}(\iota_\mu(c)g(\varepsilon)) = g(\varepsilon)\iota_\lambda^{-1}(\iota_\mu(c)g(\varepsilon))g(\varepsilon)^{-1} \to c$$

as $\varepsilon \to 0$.

Theorem 2.4 (confluence for character). *For every* $c \in H_\mu$ *and every column vector* $\beta \in \boldsymbol{C}^n$, *we have*

$$\lim_{\varepsilon\to 0} \chi_\lambda(\iota_\lambda^{-1}(\iota_\mu(c)g(\varepsilon))\,;\, g(\varepsilon)^{-1}\beta) = \chi_\mu(c\,;\,\beta).$$

Therefore, for $t \in \boldsymbol{C}^r$ *and* $z \in Z_{r,n}$, *we have*

$$\lim_{\varepsilon\to 0} \chi_\lambda(\iota_\lambda^{-1}(\langle t, zg(\varepsilon)\rangle)\,;\, g(\varepsilon)^{-1}\beta) = \chi_\mu(\iota_\mu^{-1}(\langle t, z\rangle)\,;\,\beta).$$

As an immediate consequence of Proposition 1.3 and Theorem 2.4, we obtain the following main theorem.

Theorem 2.5 (confluence for CHG system). *The system* $M_\lambda(zg(\varepsilon)\,;\, g(\varepsilon)^{-1}\beta)$ *tends to the system* $M_\mu(z\,;\,\beta)$ *as* $\varepsilon \to 0$.

Remark 2.6. If $z \in Z_{r,n}$ is a nonsingular point of the system M_μ, then $zg(\varepsilon) \in Z_{r,n}$ is also a nonsingular point of the system M_λ.

Example 2.7. Let us observe that the confluence of two singular points of the equation (0.1) stated in the introduction is a simple example of our general process of confluence. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 1, 1, 1)$ and $\mu = (\mu_1, \mu_2, \mu_3) = (2, 1, 1)$ be two compositions of 4. We consider μ to be adjacent to λ in the sense that $\mu_1 = \lambda_1 + \lambda_2$, $\mu_2 = \lambda_3$, $\mu_3 = \lambda_4$. Set

$$z(x) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -x & 1 & -1 \end{pmatrix} \in Z_{2,4}$$

$$w(\xi) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -\xi & 1 & -1 \end{pmatrix} \in Z_{2,4}.$$

Then, the equation (0.1) (or (0.3)) is the restriction of the system $M_\lambda(z\,;\,{}^t(-\gamma+\beta, -\beta, \alpha-1, \gamma-\alpha-1))$ (or $M_\mu(w\,;\,{}^t(-\gamma, -1, \alpha-1, \gamma-\alpha-1))$) on the 1-dimensional space $z = z(x)$ (or $w = w(\xi)$). Note that

$$g(\varepsilon) = g_{\lambda\to\mu}(\varepsilon) = \begin{pmatrix} \begin{matrix} 1 & 1 \\ 0 & \varepsilon \end{matrix} & O \\ O & I_2 \end{pmatrix}$$

$$w(\xi)g(\varepsilon) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -\varepsilon\xi & 1 & -1 \end{pmatrix}$$

$$g(\varepsilon)^{-1}\,{}^t(-\gamma, -1, \alpha-1, \gamma-\alpha-1) = {}^t\left(-\gamma+\frac{1}{\varepsilon}, -\frac{1}{\varepsilon}, \alpha-1, \gamma-\alpha-1\right).$$

From Theorem 2.5, it follows that

$$\lim_{\varepsilon\to 0} M_\lambda\left(wg(\varepsilon)\,;\,{}^t\left(-\gamma+\frac{1}{\varepsilon}, -\frac{1}{\varepsilon}, \alpha-1, \gamma-\alpha-1\right)\right) = M_\mu(w\,;\,{}^t(-\gamma, -1, \alpha-1, \gamma-\alpha-1)).$$

Remark that the equation (0.2) is the restriction of $M_\lambda\left(wg(\varepsilon)\,;\,{}^t\left(-\gamma+\frac{1}{\varepsilon}, -\frac{1}{\varepsilon}, \alpha-1, \gamma-\alpha-1\right)\right)$ on $w = w(\xi)$ and that

$$M_\lambda(z(x)\,;\,{}^t(-\gamma+\beta, -\beta, \alpha-1, \gamma-\alpha-1) = M_\lambda\left(w(\xi)g(\varepsilon)\,;\,{}^t\left(-\gamma+\frac{1}{\varepsilon}, -\frac{1}{\varepsilon}, \alpha-1, \gamma-\alpha-1\right)\right)$$

if and only if

$$x = \varepsilon\xi, \quad \beta = \frac{1}{\varepsilon}.$$

Thus the confluence process from (0.1) to (0.3) is equivalent to that from $M_\lambda(z(x)\,;{}^t(-\gamma+\beta, -\beta, \alpha-1, \gamma-\alpha-1))$ to $M_\mu(w(\xi)\,;\,{}^t(-\gamma, -1, \alpha-1, \gamma-\alpha-1))$.

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