85. *On the Neumann Problems for Certain Degenerate Elliptic Operators**),**)

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§1. Introduction. Let Ω be a bounded domain in $\boldsymbol{R}^n (n \geq 2)$ with $\partial\Omega \in C^\infty$. Let φ be a given nonnegative smooth function on $\bar{\Omega}$ and equivalent to a distance to the boundary:

$$\Omega = \{\varphi(x) > 0\},\ \partial\Omega = \{\varphi(x) = 0\},\ d\varphi \neq 0 \quad \text{on } \partial\Omega.$$

Let $\alpha(x)$ be a C^2-function on $\bar{\Omega}$. In the Dirichlet problem (D-P), we impose on α the following condition (D).

$$\text{(D)} \qquad \frac{d\alpha}{dn} = 0 \quad \text{on } \partial\Omega,\ \boldsymbol{n}: \text{unit normal}.$$

The purpose of this note is to study the homogeneous Dirichlet and Neumann boundary problems defined by

$$\text{(D-P)} \qquad \begin{cases} -\operatorname{div}(\varphi^\alpha \nabla u) + c(x)\varphi^{\alpha-1}u = f & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \quad \text{We assume (D).} \end{cases}$$

$$\text{(N-P)} \qquad \begin{cases} -\operatorname{div}(\varphi^\alpha \nabla u) + c(x)\varphi^{\alpha}u = f & \text{in } \Omega \\ \dfrac{du}{dn} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

As for (D-P), we have

Proposition 1. *Suppose that α satisfies* (D) *and $\alpha < 1$ on $\partial\Omega$. Then, there exists a positive number M such that if $\inf_{x\in\Omega} c(x) \geq M$, then for every $f \in C^\mu(\bar{\Omega})$, there exists one and only one solution u to* (D-P) *which is written as $u = \varphi^{1-\alpha}v$ with a function v belonging to $C^{1+\mu}(\bar{\Omega})$ such that $\varphi v \in C^{2+\mu}(\bar{\Omega})$.*

From the theory of ordinary differential equations, we see that (D-P) has no solution vanishing at $x_n = 0$ if $\alpha \geq 1$. So, the condition $\alpha < 1$ is necessary for us. If we set $u = \varphi^{1-\alpha}v$ in (D-P), this proposition follows as a corollary to the theorem due to C. Goulaouic-N. Shimakura [1] in which the equation $\varphi\Delta v + z\partial_n v + f = 0 (f \in C^\mu(\bar{\Omega}), \Re z > 0)$ were studied. Note that the restriction (D) can be relaxed as follows:

$$\alpha \in C^2(\Omega),\quad |(\nabla\varphi, \nabla\alpha)|,\ \varphi|\Delta\alpha| = O(\varphi^\delta), \text{ for some } \delta > \mu.$$

§2. Preliminaries to the Neumann problem. In the Neumann problem, we have to deal with unbounded solutions as we shall show in §4. So, we are obliged to modify the classical Schauder spaces to admit unbounded solutions.

Definition. Let $0 < \mu < 1,\ \beta \neq 0$, and let $f \in C(\boldsymbol{R}^n_+)$ be compactly

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supported in $\overline{\boldsymbol{R}_+^n}$. A function f is said to belong to the class $\Lambda_\beta^\mu(\boldsymbol{R}_+^n)$ if

$$|f|_{\mu,\beta} = \sup_{x,y} \min(x_n^{(\mu-\beta)_+}, y_n^{(\mu-\beta)_+}) \frac{|f(x)-f(y)|}{|x-y|^\mu}$$

is finite. Here $(\mu-\beta)_+ = \max(0, \mu-\beta)$. For a positive integer k, f is said to belong to the class $\Lambda_\beta^{k+\mu}(\boldsymbol{R}_+^n)$ if $D^\gamma f \in \Lambda_{\beta-k}^\mu(\boldsymbol{R}_+^n)$ for any γ with $|\gamma| = k$ and we denote

$$|f|_{k+\mu,\beta} = \sum_{|\gamma|=k} |D^\gamma f|_{\mu,\beta-k}.$$

We also set $\Lambda_\beta^\mu(\boldsymbol{R}_+^n\,;loc) = \{f \in C(\boldsymbol{R}_+^n)\,; gf \in \Lambda_\beta^\mu(\boldsymbol{R}_+^n)$, for any $g \in C_0^\infty(\overline{\boldsymbol{R}_+^n})\}$.

$|\cdot|_{\mu,\beta}$ is then a norm on $\Lambda_\beta^\mu(\boldsymbol{R}_+^n)$, and $\Lambda_\beta^\mu(\boldsymbol{R}_+^n)$ is endowed with the inductive limit topology of Banach spaces exactly as $C^\mu(\boldsymbol{R}_+^n)$. For these spases we can show

Lemma 1. (1) *$C^\mu(\boldsymbol{R}_+^n) \subset \Lambda_\beta^\mu(\boldsymbol{R}_+^n)$, and $\Lambda_\beta^\mu(\boldsymbol{R}_+^n) = C^\mu(\boldsymbol{R}_+^n)$ if $\beta \geq \mu$.*

(2) *If $f \in \Lambda_\beta^\mu(\boldsymbol{R}_+^n)$ and $\beta < 0$, then $f \in C_\beta = \{f : f \in C(\boldsymbol{R}_+^n)$ and $\sup|f|\,x_n^{-\beta} < +\infty\}$.*

(3) *If $\beta > 0$ and $f \in \Lambda_\beta^\mu(\boldsymbol{R}_+^n)$, then $f(x',\cdot) \in C^{\min(\beta,\mu)}(\boldsymbol{R}_+^n)$, uniformly in $x' = (x_1,\dots,x_{n-1}) \in \boldsymbol{R}^{n-1}$.*

(4) *$f \in \Lambda_\beta^\mu(\boldsymbol{R}_+^n) \cap C_\beta$ if and only if $f/x_n^\alpha \in \Lambda_{\beta-\alpha}^\mu(\boldsymbol{R}_+^n) \cap C_{\beta-\alpha}$.*

(5) *If $f \in \Lambda_{1+\beta}^{1+\mu}(\boldsymbol{R}_+^n)$, $(\beta > -1+\mu)$, then $f \in C^\mu$.*

(6) *$\log x_n \in \Lambda_\beta^\mu(\boldsymbol{R}_+^n\,;loc)$ if and only if $\beta < 0$.*

In (2), *the number $\sup_{\boldsymbol{R}_+^n} |f|\,x_n^{-\beta}$ is uniformly bounded for f with support in a fixed compact set.*

Our goal is the theorem of existence and uniqueness of the solution to (N-P) in the function space $\Lambda_{2+\tau}^{2+\mu}$ (Theorem 2 below). And this is relied upon an a priori inequality for solutions (Theorem 1 below) and the method of continuity.

To obtain an a priori inequality, we proceed as follows. By a partition of unity of the closure $\bar{\Omega}$, the question is essentially reduced to prove an inequality for u with small support in a neighborhood of a boundary point. This is because the operator is not degenerate in the interior of the domain so that a nice inequality in the interior is guaranteed by a classical result of J. Schauder. Let x^0 be any boundary point. Then, there exists a neighborhood U of x^0 and a diffeomorphism χ from $U \cap \bar{\Omega}$ in the x-space to a semi-ball $\{|y| < r,\ y_n \geq 0\}$ in the y-space such that $\chi(x^0) = 0$, $\chi(U \cap \partial\Omega) \subset \{y_n = 0\}$ and that $\varphi(\chi^{-1}(y_n)) = y_n$. Clearly, $\alpha(x)$ is close to the constant function $\alpha(x^0)$ in U. Rewriting the new coordinates y again by x, we can locally approximate (N-P) by a problem

$$\text{(N-P')} \qquad -x_n^\alpha \Delta u - \alpha x_n^{\alpha-1}\partial_n u = f \quad \text{in } \boldsymbol{R}_+^n\,; \frac{\partial u}{\partial x_n} = 0 \text{ on } \{x_n = 0\}$$

for u with small support in a neighborhood of the origin. Here α is a constant satisfying $\alpha > -1$ and f on the right hand side contains an error term arising from the localization.

Green function for (N-P'). For $\alpha > 0$, let us set

$$K_\alpha(x,y) = C_\alpha |x-y^*|^{-\alpha} |x-y|^{2-n} F(\alpha,\omega), \quad \alpha > 0,$$

where

$$y^* = (y', -y_n), \quad \omega = 1 - \frac{|x-y|^2}{|x-y^*|^2} = \frac{4x_n y_n}{|x-y^*|^2},$$

$$C_\alpha = 2^{\alpha-2}\pi^{-n/2}\Gamma\Big(\frac{n+\alpha-2}{2}\Big)\Gamma\Big(\frac{\alpha}{2}\Big)/\Gamma(\alpha)$$

and $F(\alpha, \omega)$ is a hypergeometric function defined by

$$F(\alpha, \omega) = \frac{\Gamma(\alpha)}{\Gamma\Big(\frac{\alpha+2-n}{2}\Big)\Gamma\Big(\frac{\alpha}{2}\Big)} \sum_{j=0}^{\infty} \frac{\Gamma\Big(j + \frac{\alpha+2-n}{2}\Big)\Gamma\Big(j+\frac{\alpha}{2}\Big)}{\Gamma(j+\alpha)} \frac{\omega^j}{j!}.$$

The kernel $K_\alpha(x, y)$ is not defined for $\alpha \leq 0$. However, it can be continued to an entire function with respect to α. For example,

$$K_0(x, y) = C_{(0)}(|x-y|^{2-n} + |x-y^*|^{2-n}),$$

$$K_{-2}(x, y) = C_{(-2)}(|x-y|^{4-n} + |x-y^*|^{4-n} - (n-4)x_n y_n |x-y|^{2-n} + (n-4)x_n y_n |x-y^*|^{2-n}), \quad \text{if} \quad n > 4.$$

Here $C_{(0)}$ and $C_{(-2)}$ are positive numbers independent of $(x, y) \in \boldsymbol{R}^n_+$.

Then

$$u = K_\alpha f = \int_{\boldsymbol{R}^n_+} K_\alpha(x, y) f(y) dy$$

is a solution of the problem (N-P'). Following is the key lemma for the Green function.

Lemma 2. *Suppose that* $\alpha > -1$, $-1 + \mu < \beta \neq 0$ *and* $f/x_n^\alpha \in \Lambda^\mu_{\beta-\alpha}(\boldsymbol{R}^n_+)$. *Let* $u = K_\alpha f$. *Then,* $\partial_j \partial_k u$, $(\partial_n u)/x_n \in \Lambda^\mu_{\beta-\alpha}(\boldsymbol{R}^n_+ ; loc)$ *for* $1 \leq j, k \leq n$. *Furthermore,* $\partial_n u = 0$ *on the boundary if* $\beta > \alpha - 1$.

§3. Results. First we define the function spaces $\Lambda^\mu_\tau(\Omega)$ analogous to the previous ones.

Definition. Let $0 < \mu < 1$, $\tau \neq 0$. A function u is said to belong to the class $\Lambda^\mu_\tau(\Omega)$ if

$$\| u \|_{\mu,\tau} = | u |_{\mu,\tau} + \sup | \varphi^{(-\tau)_+} u |$$

is finite, where

$$| u |_{\mu,\tau} = \sup_{\Omega\times\Omega} \min(\varphi(x)^{(\mu-\tau)_+}, \varphi(y)^{(\mu-\tau)_+}) \frac{| u(x) - u(y) |}{| x-y |^\mu}.$$

Also, u is said to belong to the class $\Lambda^{2+\mu}_{2+\tau}(\Omega)$ if $D^\gamma u \in \Lambda^\mu_\tau(\Omega)$ for any γ with $|\gamma| \leq 2$ and we denote $\| u \|_{2+\mu,2+\tau} = \sum_{|\gamma|\leq 2} \| \partial^\gamma u \|_{\mu,\tau}$.

Theorem 1. *Assume that* $\alpha(x) > -1$ *on* $\partial\Omega$. *Let* τ, μ *and* α *satisfy*

(1) $$\max(-1, -1 + \mu - \alpha(x)) < \tau < 0 \quad \text{on} \quad \partial\Omega.$$

Let $u \in \Lambda^{2+\mu}_{2+\tau}(\Omega)$ *be a solution of (N-P) for* $f/\varphi(x)^{\alpha(x)} \in \Lambda^\mu_\tau(\Omega)$. *Then it holds that*

$$\| u \|_{2+\mu,2+\tau} + \| \varphi^{-1}(\nabla\varphi, \nabla u) \|_{\mu,\tau} \leq C \| \frac{f}{\varphi^\alpha} \|_{\mu,\tau} + C' \| u \|_{\mu,\tau}.$$

Here C *and* C' *are positive numbers independent of* f *and* u.

We get first the inequality for solutions to (N-P') with small support with the aid of Lemma 2 and estimate the error terms arising from the partition of unity. We need Lemma 3 below for an interpolation argument and to treat the terms involving $\log \varphi$. Finally, to prove Theorem 1, we sum up the both sides of a finite number of inequalities in the local coordinate neighborhoods.

The existence and uniqueness of the solution to (N-P), which is the main theorem in this article, is stated as follows:

Theorem 2. *Assume that $\alpha(x) > -1$ on $\partial\Omega$, $c(x) > 0$ on $\bar{\Omega}$ and that*

$$(2)\max\left(-1+\mu,\ -1+\mu-\alpha(x),\ -\frac{1}{2},\ -\frac{\alpha(x)+1}{2}\right) < \tau < 0 \text{ on } \partial\Omega.$$

Then, (N-P) has a unique solution in the space $\Lambda^{2+\mu}_{2+\tau}(\Omega)$ for every f such that $f/\varphi^{\alpha} \in \Lambda^{\mu}_{\tau}(\Omega)$.

Theorem 2 is proved in the following way: First, we prove the uniqueness of the solution with the aid of a function space

$$W(\Omega) = \left\{u : \int_{\Omega}(|\nabla u|^2 + c\,|u|^2)\varphi^{\alpha}\,dx < \infty\right\}.$$

Note that f belongs to the dual space of $W(\Omega)$ provided that $\tau > -(\alpha+1)/2$ and $\alpha > -1$ on $\partial\Omega$. Since the weak solution is unique in the space $W(\Omega)$, the classical solution belonging to $\Lambda^{2+\mu}_{2+\tau}(\Omega)$ is also unique if there exists any. Second, if $\alpha(x)$ is identically equal to 0, then (N-P) is a variant of the classical Neumann problem for the Laplacian in a bounded domain. Third, for general α, we make use of the method of continuity. More precisely, we introduce a family of operators

$$L_{\theta}u = -\varphi^{-\theta\alpha}\operatorname{div}(\varphi^{\theta\alpha}\operatorname{grad} u) + cu$$

with parameter $\theta \in [0, 1]$. The given equation is then $L_1 u = f/\varphi^{\alpha}$. Let F be the set of θ such that L_{θ} has the continuous inverse operator under the homogeneous Neumann condition. Then, F is non-empty because $0 \in F$ by virtue of the above remark. And Theorem 1 guarantees that F is an open and closed subset of [0, 1]. Therefore, $F = [0, 1]$, in particular, $1 \in F$. Consequently, (N-P) has a unique solution for every f satisfying the prescribed condition. In each step of the reasoning, we need some restrictions on the values of $c(x)$, $\alpha(x)$, μ and τ (see (1) and (2)).

Remark. Theorems 1 and 2 are also valid for the following (N-P") if $\tau \neq 0$ (in place of $\tau < 0$) and if $\inf_{\Omega} c(x)$ is sufficiently large.

$$\text{(N-P")}\qquad \begin{cases} -\varphi^{\alpha}\Delta u - \alpha\varphi^{\alpha-1}(\nabla\varphi,\ \nabla u) + c(x)\varphi^{\alpha}u = f \text{ in } \Omega \\ \dfrac{du}{dn} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

In the proof of Theorem 1, we need the following interporlation lemma:

Lemma 3. (1) *Assume that $u \in \Lambda^{2+\mu}_{2+\tau}(\Omega)$, $\tau > -1+\mu$. Then*

$$\sum_{|r|\leq 1} \|\partial^r u\|_{\mu,\mu} \leq C(\Omega)\left[\sum_{|r|=2} \|\partial^r u\|_{\mu,\tau} + \|u\|_{\mu,\tau}\right].$$

(2) *Let $p \in \partial\Omega$, $w_R = B_R(p) \cap \Omega$ for a small $R > 0$ and $u \in \Lambda^{2+\mu}_{2+\tau}(w_R)$, $0 > \tau > -1+\mu$. Then*

$$\|\partial_j u \log\varphi\,;\Lambda^{\mu}_{\tau}(w_R)\| \leq O(R^{-\tau}|\log R|)\,\|u\|_{2+\mu,2+\tau}, \text{ for } j = 1,2,\dots,n.$$

§4. Example. Here we give an example of unbounded solution to (N-P). For simplicity we consider (N-P) locally in some neighborhood of the origin of $\boldsymbol{R}^n_+$ in place of a general domain Ω.

Example. *Let L be the operator defined to be*

$$Lu = -\operatorname{div}(x_n^{x_1}\nabla u) = -x_n^{x_1}\left[\Delta u + \frac{x_1}{x_n}\partial_n u + (\log x_n)\partial_1 u\right].$$

Then, for $u = x_n^2\log x_n - 2x_1 - 2x_1^2$, we have $\dfrac{L_u}{x_n^{x_1}} = -1 + x_1 \in C^\mu$ and $\dfrac{\partial u}{\partial x_n}\Big|_{x_n=0} = 0$. However $\partial_n^2 u = 3 + 2\log x_n \notin C^\mu$. And we see that $u \in \Lambda^{2+\mu}_{2+\tau}(\boldsymbol{R}^n_+\,;\,loc)$ for any $\mu \in (0,1)$ and any $\tau < 0$.

Note also that $w = x_n^{1-x_1}$ satisfies $Lw = 0$ with the homogeneous Dirichret condition $w|_{x_n=0} = 0$ if $x_1 < 1$ and the homogeneous Neumann condition $\dfrac{\partial w}{\partial x_n}\Big|_{x_n=0} = 0$ if $x_1 < 0$ respectively. According to Proposition 1, we can consider w as a null-solution to the Dirichret problem (see (D-P) with $c = 0$). Moreover, we see that w is excluded from our framework for the problem (N-P). To see this, let K be a compact set contained in $\overline{\boldsymbol{R}^n_+} \cap \{x : -1 < x_1 < 0\}$ such that $K \cap \{x : x_n = 0\} \neq \phi$ and assume that $w \in \Lambda^{2+\mu}_{2+\tau}(K\,;\,loc)$ for some $\mu \in (0,1)$ and $\tau < 0$. Then, we have $\dfrac{\partial^2 w}{\partial x_n^2} = x_1(x_1 - 1)x_n^{-1-x_1} \in \Lambda^\mu_\tau(K\,;\,loc)$. Furthermore, it follows from the assertion (2) in Lemma 1 that $\tau \leq \min_{x\in K\cap\{x_n=0\}}(-1-x_1)$. But this contradicts to the conditions (1) and (2) in the main theorems, so that the assertion follows.

References

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