80. The Schur Indices of the Irreducible Characters of $G_2(2^n)$

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Introduction. Let $G_2(q)$ be the finite Chevalley group of type (G_2) over a finite field $\boldsymbol{F}_q$ with q elements. It was shown in [3] that the following theorem holds for odd q:

Theorem. *The Schur index $m_{\boldsymbol{Q}}(\chi)$ of any complex irreducible character χ of $G_2(q)$ with respect to $\boldsymbol{Q}$ is equal to* 1.

In this paper, we shall prove that the theorem holds also for $q = 2^n$, as was announced in [3]. The complex irreducible characters of $G_2(2^n)$ have been calculated by the first named author and H. Yamada in [2]. In the following, $G_2(2^n)$ will be denoted simply by G.

Proof of the theorem for $q = 2^n$. For the notation of the conjugacy classes of $G = G_2(2^n)$, the characters of G, or of subgroups of G, etc., we follow those in [2].

Let B be the Borel subgroup of G and U its unipotent part. We first describe the character-values of the Gelfand-Graev character Γ_G of G and the induced character $1_U^G = \mathrm{Ind}_U^G(1_U)$; Γ_G is the character of G induced by the linear character of U given by $x_a(t_1)x_b(t_2)x_{a+b}(t_3)\cdots x_{3a+2b}(t_6) \to \phi(t_1)\phi(t_2)$, where ϕ is a previously fixed non-trivial additive character of $\boldsymbol{F}_{2^n}$. There are eight unipotent classes in $G : A_0, A_1, A_2, A_{31}, A_{32}, A_4, A_{51}$ and A_{52}; representatives of these classes are respectively: $h(1, 1, 1) = e$, $x_{3a+2b}(1)$, $x_{2a+b}(1)$, $x_{a+b}(1)x_{2a+b}(1)$, $x_{a+b}(1)x_{2a+b}(1)x_{3a+b}(\xi)$, $x_b(1)x_{2a+b}(1)x_{3a+b}(\eta)$, $x_a(1)x_b(1)$ and $x_a(1)x_b(1)x_{2a+b}(\xi)$. Then we have the following table:

	Γ_G	1_U^G
A_0	$(q^2-1)(q^6-1)$	$(q^2-1)(q^6-1)$
A_1	$-q^2+1$	$(q^2-1)(q^3-1)$
A_2	$-q^2+1$	$(q^2-1)(q-1)(2q+1)$
A_{31}	$-q^2+1$	$(q-1)^2(4q+1)$
A_{32}	$-q^2+1$	$(q-1)^2(2q+1)$
A_4	$-q^2+1$	$(q^2-1)(q-1)$
A_{51}	1	$(q-1)^2$
A_{52}	1	$(q-1)^2$

Since ϕ takes values in $\boldsymbol{Q}$, Γ_G is realizable in $\boldsymbol{Q}$. Also 1_U^G is clearly realizable in $\boldsymbol{Q}$.

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Now let us recall a part of Schur's theorem:

Lemma (see, e.g., [1], 11.4). *Let μ be an ordinary character of a finite group H which is realizable in a field K of characteristic* 0. *Then, for any irreducible character χ of H, the Schur index $m_K(\chi)$ of χ with respect to K divides the inner product $\langle\chi, \mu\rangle_H$.*

We now prove the theorem in Introduction. By using [2] and the information about the values of Γ_G above, we find that the irreducible constituents of Γ_G are precisely those irreducible characters of G of degree, as a polynomial in $q=2^n$, six, and that Γ_G is multiplicity-free. Then, as Γ_G is realizable in $\boldsymbol{Q}$, by the lemma, we find that $m_{\boldsymbol{Q}}(\chi)=1$ for any irreducible character χ of G such that $\chi(1)$ is of degree six. Next we have $\langle\theta_i, 1_U^G\rangle_G=1$ for $i=3, 4$, hence we have $m_{\boldsymbol{Q}}(\theta_3)=m_{\boldsymbol{Q}}(\theta_4)=1$.

Let $\theta_4(\pm 1, \pm 1)$ be four irreducible characters of B constructed in [2], p. 334; in view of [2], p. 333, we find that the characters $\theta_4(\pm 1, \pm 1)$ are realizable in $\boldsymbol{Q}$; hence the $\theta_4(\pm 1, \pm 1)^G$ are realizable in $\boldsymbol{Q}$. We have: $\langle\theta_1 | B, \theta_4(1, 1)\rangle_B=\langle\theta_1' | B, \theta_4(-1, 1)\rangle_B=\langle\theta_2 | B, \theta_4(1, -1)\rangle_B=\langle\theta_2' | B, \theta_4(-1, -1)\rangle_B=1$. To show these equalities, we use the following correspondence between classes of B and those of G :

Classes in B	A_0	A_1	A_2	A_3	A_{41}	A_{42}	A_{43}	A_{51}	$A_{52}(0)$
Classes in G	A_0	A_1	A_1	A_2	A_2	A_{31}	A_{32}	A_1	$A_{32}(\varepsilon=-1)$

Classes in B	$A_{52}(0)$	$A_{52}(i)\ (i\neq 0)$	$A_{53}(0)$	$A_{53}(t)\ (t\in\Omega_1)$
Classes in G	$A_{31}(\varepsilon=1)$	A_4	A_2	A_{32}

Classes in B	$A_{53}(t)\ (t\in\Omega_2)$	$A_{53}(t)\ (t\in\Omega_3)$	A_{61}	A_{62}	A_{63}
Classes in G	A_{31}	A_4	A_2	A_{31}	A_{32}

Classes in B	B_{71}	A_{72}
Classes in G	A_{51}	A_{52}

$(\varepsilon=(-1)^n)$.

Hence, by the Frobenius-reciprocity and the lemma above, we have $m_{\boldsymbol{Q}}(\theta_1)=m_{\boldsymbol{Q}}(\theta_1')=m_{\boldsymbol{Q}}(\theta_2)=m_{\boldsymbol{Q}}(\theta_2')=1$.

The irreducible characters of G except for two characters $\theta_9(1)$, $\theta_9(2)$ are real, so that, by the Brauer-Speiser theorem (see [4], p. 9), all the irreducible characters of G except for $\theta_9(1)$, $\theta_9(2)$ have the Schur indices at most 2 over $\boldsymbol{Q}$. Thus any irreducible character of G of odd degree has the index 1 over $\boldsymbol{Q}$. The remaining characters are θ_8, $\theta_9(1)$, $\theta_9(2)$.

Let θ_2 be the irreducible character of B constructed in [2], p. 332; θ_2 is realizabie in $\boldsymbol{Q}$. We have $\langle\theta_8 | B, \theta_2\rangle_B=q+\varepsilon\not\equiv 0 \pmod 2$, hence we have $m_{\boldsymbol{Q}}(\theta_8)\not\equiv 0 \pmod 2$ and $m_{\boldsymbol{Q}}(\theta_8)=1$.

Finally, let $\chi=\theta_9(1)$ or $\theta_9(2)$. Let ζ_3 be a primitive cubic root of unity. Then we have $\boldsymbol{Q}(\chi)=\boldsymbol{Q}(\zeta_3)$, where $\boldsymbol{Q}(\chi)$ is the field generated over $\boldsymbol{Q}$ by the values of χ. According to Proposition 1 of [3], for any non-regular unipotent element u of G, $\chi(u)$ is a rational integer and $m_{\boldsymbol{Q}}(\chi) \mid \chi(u)$. We find from Table IV-2 of [2], p. 366, that $\chi(A_{31})=\frac{1}{3}q(\varepsilon q-1)$ and $\chi(A_4)=\frac{1}{3}q(\varepsilon q+2)$. Hence $m_{\boldsymbol{Q}}(\chi) \mid q\ (=2^n)$. We show that $m_{\boldsymbol{Q}}(\chi)\not\equiv 0 \pmod 2$. Let t be an element of order 3 whose class is called B_0 in [2]. Let μ be a non-trivial linear character of $\langle t\rangle$. Then we have

$$\langle \chi \mid \langle t\rangle, \mu\rangle_{\langle t\rangle} = \frac{1}{9}(q^2-1)(q^3-\varepsilon) \not\equiv 0 \pmod 2.$$

And $\boldsymbol{Q}(\chi)=\boldsymbol{Q}(\mu)=\boldsymbol{Q}(\zeta_3)$. Thus we have $m_{\boldsymbol{Q}}(\chi) \not\equiv 0 \pmod 2$ and $m_{\boldsymbol{Q}}(\chi) = 1$. This completes the proof of our theorem.

References

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