74. *Some Remarks on the Class of Riemann Surfaces with (W)-property*

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Introduction. In the classification theory we know that some classes of Riemann surfaces are characterized in terms of the subspaces of real square integrable harmonic differentials. For example, $\Gamma_{he}(R) \cap {}^*\Gamma_{he}(R) = \{0\}$ (resp. $\Gamma_{he}(R) \cap {}^*\Gamma_{hse}(R) = \{0\}$) if and only if $R \in O_{AD}$(resp. $R \in O_{KD}$). (See §1 for notations.) In the papers [3,6] M. Watanabe (neeé Mori) introduced the following condition, which we call here (W)-property,

$$\Gamma_{he}(R) \cap {}^*\Gamma_{hse}(R) \subset {}^*\Gamma_{he}(R)$$

or equivalently

$$\Gamma_{ho}(R) \cap {}^*\Gamma_{ho}(R) = \Gamma_{hse}(R) \cap {}^*\Gamma_{ho}(R)$$

for a Riemann surface R. She obtained other equivalent conditions and interesting conseqences.

In the paper [1] we have given a new characterization of (W)-property in terms of specific period reproducing differentials.

In the present paper we shall consider the class of Riemann surfaces with (W)-property, which we denote by P_W, in the context of the classification theory.

1. Preliminaries. For the sake of convenience we recall some definitions. Let $\Gamma_h(R)$ be the Hilbert space of real square integrable harmonic differentials on a Riemann surface R, where the inner product is given by

$$(\omega_1, \omega_2) = (\omega_1, \omega_2)_R = \int\int_R \omega_1 \wedge {}^*\omega_2,$$

${}^*\omega_2$ being the conjugate differential of ω_2. Let $\Gamma_{he}(R)$ (resp. $\Gamma_{hse}(R)$) be the subspace of $\Gamma_h(R)$ whose elements ω are exact (resp. semiexact) on R, that is

$$\int_\gamma \omega = 0 \text{ for every (resp. every dividing) 1-cycle } \gamma \text{ on } R.$$

Given a closed subspace Γ_y of Γ_h, the orthogonal complement of Γ_y in Γ_h is denoted by $\Gamma_y^\perp$. For the spaces $\Gamma_{ho} = ({}^*\Gamma_{he})^\perp$ and $\Gamma_{hm} = ({}^*\Gamma_{hse})^\perp$, the following inclusion relations hold

$$\Gamma_h \supset \Gamma_{hse} \supset \Gamma_{he} \supset \Gamma_{hm}\,;\, \Gamma_{hse} \supset \Gamma_{ho} \supset \Gamma_{hm}.$$

The Γ_{hm} is known as the space of harmonic measure differentials. For a given 1-cycle c on R and a closed subspace Γ_y of Γ_h there exists uniquely the period reproducing differential $\sigma_y(c)$ in Γ_y such that

$$\int_c \omega = (\omega, \sigma_y(c))_R \text{ for every } \omega \in \Gamma_y.$$

We are interested in $\sigma_{hse}(c)$ and $\sigma_{ho}(c)$.

Let $HD(R)$ be the class of real-valued harmonic functions on R with finite Dirichlet integral and $ReAD(R)$ (resp. $KD(R)$) be the subclass of $HD(R)$ whose elements u have the following property;

$$\int_{\gamma} {}^{*}du = 0 \text{ for every (resp. every dividing) 1-cycle } \gamma \text{ on } R$$

where ${}^{*}du$ is the conjugate differential of du. We know that

$$\{du\,;\, u \in HD(R)\} = \Gamma_{he}(R)$$
$$\{du\,;\, u \in KD(R)\} = \Gamma_{he}(R) \cap {}^{*}\Gamma_{hse}(R)$$
$$\{du\,;\, u \in ReAD(R)\} = \Gamma_{he}(R) \cap {}^{*}\Gamma_{he}(R).$$

We say a Riemann surface R belongs to the class O_{AD} (resp. O_{KD}) if and only if $ReAD(R)$ (resp. $KD(R)$) implies only constant functions.

2. The class P_W. 2.1 Results. Our result in [1] is the following theorem.

Theorem A. *Let R be an arbitrary Riemann surface, and $\sigma_{hse}(c)$ and $\sigma_{ho}(c)$ as above for an 1-cycle c on R.*

Then the following properties are equivalent;

(I) *R has (W)-property.*

(II) $\| \sigma_{hse}(c) \|_R = \| \sigma_{ho}(c) \|_R$ *(equivalently $\sigma_{hse}(c) = \sigma_{ho}(c)$) for every 1-cycle c on R, where $\| \omega \|_R$ is the Dirichlet norm of $\omega \in \Gamma_h(R)$.*

Furthermore, if R is of finite positive genus, the next properties are also equivalent;

(III) *R belongs to the class O_{AD}.*

(IV) $\| \sigma_{hse}(c) \|_R = \| \sigma_{ho}(c) \|_R$ *(equivalently $\sigma_{hse}(c) = \sigma_{ho}(c)$) for some non-dividing 1-cycle c on R.*

By Theorem A we know that $O_{AD} = O_{KD} = P_W$ holds for Riemann surfaces of finite genus. In case of infinite genus we show the next proposition;

Proposition 1. *For Riemann surfaces of infinite genus,*

$$O_{KD} = O_{AD} \cap P_W$$

holds and there is no inclusion relation between O_{AD} and P_W.

If R is of finite positive genus, the condition (IV) in Theorem A is equivalent to (W)-property. But in infinite case the condition (IV) is not sufficient, that is;

Proposition 2. *There exists a Riemann surface R of infinite genus on which there are non-dividing 1-cycles c_1 and c_2 having next properties;*

$$\| \sigma_{hse}(c_1) \|_R = \| \sigma_{ho}(c_1) \|_R, \ \| \sigma_{hse}(c_2) \|_R \neq \| \sigma_{ho}(c_2) \|_R.$$

For Riemann surfaces of finite genus $P_W = O_{KD}$ is quasiconformally invariant.

We show that this property is not valid in case of infinite genus.

Proposition 3. *The class P_W is not quasiconformally invariant.*

2.2. Proofs. *Proof of Proposition* 1. If a Riemann surface R belongs to O_{AD} and P_W, then $\Gamma_{he} \cap {}^{*}\Gamma_{he} = \{0\}$ and $\Gamma_{he} \cap {}^{*}\Gamma_{hse} \subset {}^{*}\Gamma_{he}$ holds for R. Therefore $\Gamma_{he} \cap {}^{*}\Gamma_{hse} = \{0\}$ and R belongs to O_{KD}.

Conversely if R belongs to O_{KD}, then

$$\Gamma_{he} \cap {}^{*}\Gamma_{he} \subset \Gamma_{he} \cap {}^{*}\Gamma_{hse} = \{0\} \subset {}^{*}\Gamma_{he}.$$

Hence $R \in O_{AD} \cap P_W$.

We know that the class O_{KD} is a proper subset of the class O_{AD} (cf.

Sario Nakai [5] Theorem II 15D, I 10B Myrberg's example).

We construct a Riemann surface which belongs to $P_W \setminus O_{AD}$.

We recall Sakai's example,"Example 1.5" in [4]. For the sake of convenience we reconstruct Sakai's example and we shall show that this Riemann surface belongs to $P_W \setminus O_{AD}$.

Example 1 (Example 1.5 [4]). Let U be the unit disc. Set

$$l_{n,m} = \left\{z = re^{i\theta}; 1 - \frac{1}{2^n} \leq r \leq \left(1 - \frac{1}{2^n}\right) + \frac{1}{2^{n+2}},\ \theta = \frac{2\pi}{[8\pi(2^n - 1)]} m\right\}$$

$$(n = 1,2,\ldots,; m = 1,2,\ldots,[8\pi(2^n - 1)])$$

where [ ] denotes Gauss' symbol. Let $U_i (i = 1, 2)$ be two copies of $U \setminus \cup_{n,m} l_{n,m}$, and join U_1 with U_2 crosswisely along every slit $l_{n,m}$. This gives a two sheeted ramified covering surface R_1 of U with a natural projection map π_1 of R_1 onto U. It is easily seen that R_1 does not belong to the class O_{AD}. It has been shown in [4] that if $\pi_1(p) = \pi_1(q)$, then $u(p) = u(q)$ for every $u \in HD(R_1)$. In other words we can identify $HD(R_1)$ with $HD(U)$ by the projection map π_1, that is for every $u \in HD(R_1)$ there exists $\tilde{u} \in HD(U)$ such that $u(p) = \tilde{u} \circ \pi_1(p)$.

Let c be an arbitrary 1-cycle on R_1.

Since $\pi_1(c)$, the projection of c, is also an 1-cycle on U, we obtain

$$\int_c {}^*du = \int_{\pi_1(c)} {}^*d\tilde{u} = 0$$

for every $u \in HD(R_1)$. This implies that $\Gamma_{he}(R_1) = \Gamma_{he}(R_1) \cap {}^*\Gamma_{he}(R_1)$ holds on R_1. Therefore R_1 has (W)-property, and we have shown that $R_1 \in P_W \setminus O_{AD}$.

To prove Proposition 2 the following lemma is needed.

Lemma 1 [1, Lemma 2]. *Let R be a Riemann surface and c be a non-dividing 1-cycle. Then* $\| \sigma_{hse}(c) \|_R = \| \sigma_{ho}(c) \|_R$ *if and only if* $\int_c {}^*du = 0$ *holds for every* $u \in KD(R)$.

Proof of Proposition 2. We construct an example of a Riemann surface R_2 on which there exist non-dividing 1-cycles c_1 and c_2 with the property

$$\| \sigma_{hse}(c_1) \|_{R_2} = \| \sigma_{ho}(c_1) \|_{R_2},\ \| \sigma_{hse}(c_2) \|_{R_2} \neq \| \sigma_{ho}(c_2) \|_{R_2}.$$

Example 2. We use the same notations as in Example 1. Let l_+(resp. l_-) be a closed interval $\left[\frac{1}{6}, \frac{1}{3}\right]\left(\text{resp. } \left[\frac{-1}{3}, \frac{-1}{6}\right]\right)$ on U. Let U' be a Riemann surface constructed by identifying the upper edge of the slit l_+ with the lower edge of the slit l_- and vice versa on $U \setminus (l_+ \cup l_-)$. Let $U_i' (i = 1, 2)$ be two copies of U' and join U_1' with U_2' crosswisely along every slit $l_{n,m}$. This gives a two sheeted ramified covering surface R_2 of U' with a natural projection map π_2 of R_2 onto U'. Then using the same arguments in the proof of Example 1.5 in [4], we can show that if $\pi_2(p) = \pi_2(q)$, then $u(p) = u(q)$ for every $u \in HD(R_2)$. Hence we can identify $HD(R_2)$ with $HD(U')$, that is for every $u \in HD(R_2)$ there exists $\tilde{u} \in HD(U')$ such that $u(p) = \tilde{u} \circ \pi_2(p)$. Since R_2 has just one ideal boundary component, it holds that $HD(R_2) = KD(R_2)$. By the same reason $HD(U') = KD(U')$.

Now we define non-dividing 1-cycles c_1, c_2 as follows

$$c_1 = \left\{|z| = \frac{11}{16}\right\} \text{ on } U_1'$$

$$c_2 = \left\{\left|z + \frac{1}{8}\right| = \frac{1}{8}, \Im z \geq 0\right\} \cup \left\{\left|z - \frac{1}{8}\right| = \frac{1}{8}, \Im z \leq 0\right\} \text{ on } U_1'.$$

First we show that $\| \sigma_{hse}(c_1) \|_{R_2} = \| \sigma_{ho}(c_1) \|_{R_2}$. By Lemma 1 it suffices to show that $\int_{c_1} {}^*du = 0$ holds for every $u \in KD(R_2)$. Since $\pi_2(c_1)$ is a dividing 1-cycle on U', for every $u \in KD(R_2) = HD(R_2)$

$$\int_{c_1} {}^*du = \int_{\pi_2(c_1)} {}^*d\tilde{u} = 0$$

holds. Hence we obtain that $\| \sigma_{hse}(c_1) \|_{R_2} = \| \sigma_{ho}(c_1) \|_{R_2}$.

To prove $\| \sigma_{hse}(c_2) \|_{R_2} \neq \| \sigma_{ho}(c_2) \|_{R_2}$ we show that there exists $u_0 \in KD(R_2)$ such that $\int_{c_2} {}^*du_0 \neq 0$. We know that the harmonic function $\tilde{u}_0(z) = y$, where $z = x + iy \in U \setminus (l_+ \cup l_-)$, can be extended to the harmonic function on U' again denoted by $\tilde{u}_0$. Set $u_0 = \tilde{u}_0 \circ \pi_2$. It is easily seen that $u_0 \in KD(R_2)$ and

$$\int_{c_2} {}^*d\tilde{u}_0 = \int_{\pi_2(c_2)} {}^*d\tilde{u}_0 = \frac{1}{z}.$$

Remark. Marden [2] considered the condition $\| \sigma_h(c) \|_R = \| \sigma_{ho}(c) \|_R$ and he proposed an open problem to construct a Riemann surface on which there exist non-dividing 1-cycles c_1 and c_2 with the condition ;

$$\| \sigma_h(c_1) \|_R = \| \sigma_{ho}(c_1) \|_R \text{ and } \| \sigma_h(c_2) \|_R \neq \| \sigma_{ho}(c_2) \|_R.$$

Since the Riemann surface R_2 of Example 2 has only one ideal boundary component, $\Gamma_h(R) = \Gamma_{hse}(R)$. Hence this Riemann surface R_2 of Example 2 is an answer to Marden's problem.

Proof of Proposition 3. We construct quasiconformally equivalent two Riemann surfaces, one belongs to the class P_W and the other does not belong to P_W.

Example 3. Let U_1, $l_\pm$ and $l_{n,m}$ the same as in Examples 1 and 2. Set

$$U_1^\star = U \setminus l_+ \setminus \bigcup_{n,m} l_{n,m} \quad U_1^{\star\star} = U_1^\star$$

$$U_2^\star = U_1^\star \quad U_2^{\star\star} = U \setminus l_- \setminus \bigcup_{n,m} l_{n,m}.$$

We obtain a Riemann surface R_3 by joining $U_1^\star$ with $U_2^\star$ crosswisely along every slit $l_{n,m}$ and l_+. This Riemann surface R_3 belongs to the class P_W (cf. Example 1). We joint $U_1^{\star\star}$ and $U_2^{\star\star}$ identifying the upper edge of the slit l_+ of $U_1^{\star\star}$ with the lower edge of the slit l_- of $U_2^{\star\star}$ and vice versa, and every common $l_{n,m}$ similarly. This gives a Riemann surface R_4. We can easily find out a quasiconformal mapping of R_3 onto R_4.

Now it suffices to show that there exist a non-dividing 1-cycle c_4 and a harmonic function $v_0 \in KD(R_4)$ such that $\int_{c_4} {}^*dv_0 \neq 0$.

Let L_1 and L_2 be closed intervals $\left[\frac{1}{3}, \frac{1}{2}\right]$ and $\left[\frac{-1}{6}, \frac{1}{2}\right]$ on $U_2^{\star\star}$. Then $c_4 = L_1 \cup L_2$ is a non-dividing 1-cycle on R_4.

Set $v_0(z) = y$ for $z = x + iy \in U_1^{\star\star} \cup U_2^{\star\star}$. This function v_0 can be extended to be harmonic on R_4, again denoted by v_0. It is easily seen that v_0 belongs to the class $KD(R_4)$. And we have

$$\int_{c_4} {}^*dv_0 = -\int_{\frac{1}{3}}^{\frac{1}{2}} dx + \int_{-\frac{1}{6}}^{\frac{1}{2}} dx = \frac{1}{2} \neq 0.$$

This completes the proof.

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